# Information Percolation in Large Markets

from work by
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# Information Percolation

- Markets
  - Hayek (1945)
  - Wolinsky (1990)
  - Golosov, Lorenzoni, Tsyvinski (2008)

- Social learning
  - Banerjee and Fudenberg (1995)
  - Acemoglu (2008)

# Setting

From Duffie and Manso (AER 2007):

- A continuum of agents matched pairwise independently to other agents at mean rate $r$.
- Payoff relevant states: $X = \begin{cases} H & \text{with probability } \nu \\ L & \text{with probability } 1-\nu \end{cases}$
- Agent $k$ is endowed with $S_k = \{s_1, \dots, s_{k_n}\}$, $\{0,1\}$-signals that are $X$-conditionally independent, with

$$P(s_i = 1 \mid X = H) \geq P(s_i = 1 \mid X = L).$$

- For almost every pair $j$ and $k$ of agents, $S_j$ and $S_k$ are disjoint.
- If $j$ and $k$ are matched, they share endowed and previously gathered signals.

# Information is Additive in Types

- For any conditional probability $p \in (0,1)$ of the event $\{X = H\}$, we define the associated information type

$$\Theta(p) = \log \frac{(1-p)\nu}{(1-\nu)p}$$

- Result: Sharing information is additive in types. That is, whenever agents of types $\theta$ and $\phi$ meet, both become type $\theta + \phi$. This process is inductive over successive matching.

# Setting for Information Percolation

**Intuition:** If the cross-sectional distribution of types is discrete, then the rate at which new agents of type $\theta$ are created is

$$2r \int \mu_t(\theta - z)\mu_t(z)\, dz = 2r(\mu_t * \mu_t)(\theta) a.s.$$

This sort of application of the LLN for random matching is known as the Stosszahlansatz (Boltzmann), and has been shown rigorously only in discrete time (Duffie and Sun, *AAP*, 2007).

# Solution for Cross-Sectional Distribution of Information

- The Boltzmann equation for the cross-sectional distribution $\mu_t$ of types is, for $\lambda = 2r$,

$$\frac{d}{dt}\,\mu_t = -\lambda\,\mu_t + \lambda\,\mu_t * \mu_t. \qquad (1)$$

- **Standing assumption:** On the event $\{X = H\}$, the first moment of $\mu_0$ is strictly positive, and $\mu_0$ has a moment generating function $z \mapsto \int e^{z\theta}\,\mu_0(d\theta)$ that is finite on a neighborhood of $z = 0$.

- **Proposition (DGM, 2008).** The unique solution of (1) is the Wild sum

$$\mu_t = \sum_{n \geq 1} e^{-\lambda t}(1 - e^{-\lambda t})^{n-1} \mu_0^{*n}. \qquad (2)$$

# Sketch of Proof of Wild Sum

The ODE for the characteristic function $\varphi(\,\cdot\,, t)$ of $\mu_t$,

$$\frac{\partial \varphi(s,t)}{\partial t} = -\lambda \varphi(s,t) + \lambda \varphi^2(s,t),$$

is solved by

$$\varphi(s,t) = \frac{\varphi(s,0)}{e^{\lambda t}(1-\varphi(s,0)) + \varphi(s,0)}.$$

This solution can be expanded as

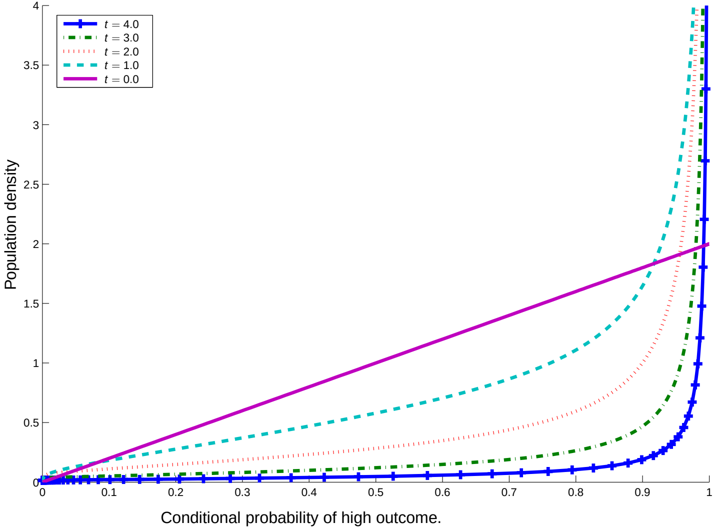
$$\varphi(s,t) = \sum_{n \geq 1} e^{-\lambda t}(1-e^{-\lambda t})^{n-1} \varphi^n(s,t),$$

which is identical to the Fourier transform of the Wild sum (2).

# Market Example

- Uninformed buyers of a contract promising $X$ randomly select two informed sellers at intensity $\lambda$.
- A second-price auction allocates the trade to the lowest-bidding seller. (The Wallet Game.)
- In the unique symmetric equilibrium, sellers bid their posterior probabilities that $X$ is high, revealing their types.

On the event $\{X = H\}$, the evolution of the cross-sectional population density of posterior probabilities of the event $\{X = H\}$.



# Convergence Rate of Population Information

Let $\pi_t$ be the cross-sectional distribution of posteriors at time $t$.

**Definition:** The rate of convergence of $\pi_t$ to $\pi_\infty$ is $\alpha > 0$ if there are constants $\kappa_0$ and $\kappa_1$ such that, for any $b$ in $(0, 1)$,

$$e^{-\alpha t}\kappa_0 \leq |\pi_t(0, b) - \pi_\infty(0, b)| \leq e^{-\alpha t}\kappa_1.$$

**Proposition:** $\pi_t$ converges at rate $\lambda$ to $\delta_0$ on the event $\{X = L\}$ and to $\delta_1$ on $\{X = H\}$.

# Meetings of More than Two at a Time

- Groups of $m$ agents are randomly matched. Because each agent is matched to others at rate $r$, the total annual quantity of attendance at meetings is $\lambda = mr$ a.s.
- The associated Boltzmann equation for the type distribution is

$$\frac{d}{dt}\mu_t = -\lambda\mu_t + \lambda\,\mu_t^{*m}.$$

- The solution is explicit as a Wild sum.

# Wild Summation Solution

The unique solution of the Boltzmann equation for $m$-at-a-time matching is

$$\mu_t = \sum_{n \geq 1} a_{(m-1)(n-1)+1} e^{-\lambda t}(1 - e^{-(m-1)\lambda t})^{n-1} \mu_0^{*[(m-1)(n-1)+1]},$$

where $a_1 = 1$ and, for $n > 1$,

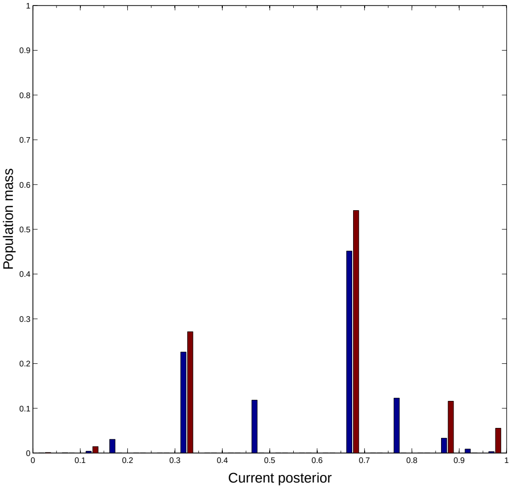
$$a_{(m-1)(n-1)+1} = \frac{1}{m-1} \left( 1 - \sum_{\left\{ \substack{i_1, \ldots, i_{(m-1)} < n \\ \sum i_k = n+m-2} \right\}} \prod_{k=1}^{m-1} a_{(m-1)(i_k - 1)+1} \right).$$

# Invariance of Convergence Rate to Group Size for a Given Total Rate of Meeting Attendance

**Proposition:** For any group size $m$, the cross-sectional distribution $\pi_t$ of posteriors converges at rate $\lambda$.

Malamud (2008) has extended this result to the case of groups of a random size.

# Groups of 2 (blue) versus Groups of 3 (red)



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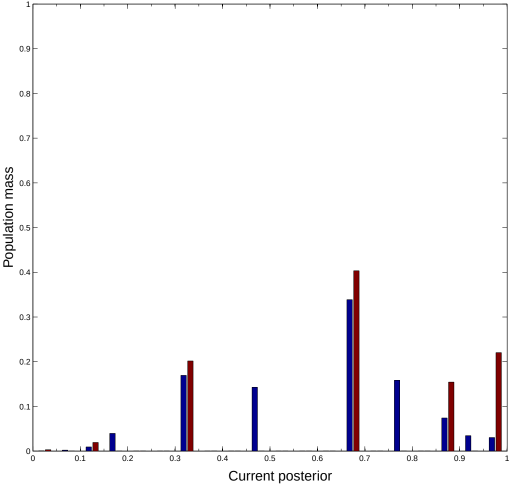

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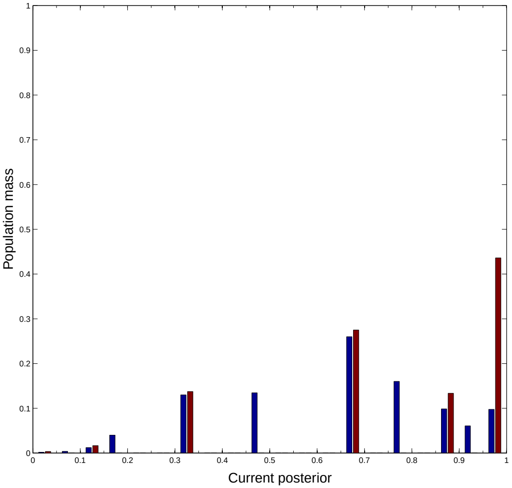

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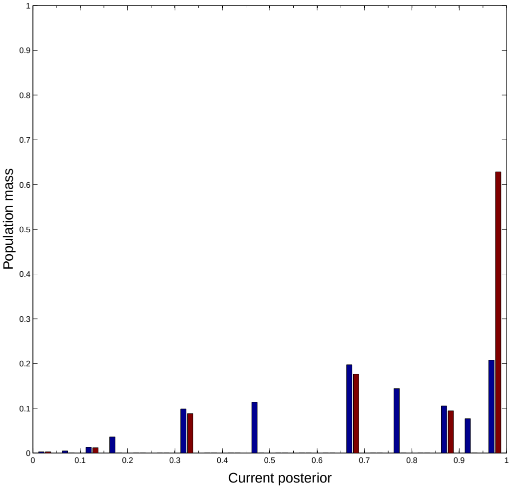

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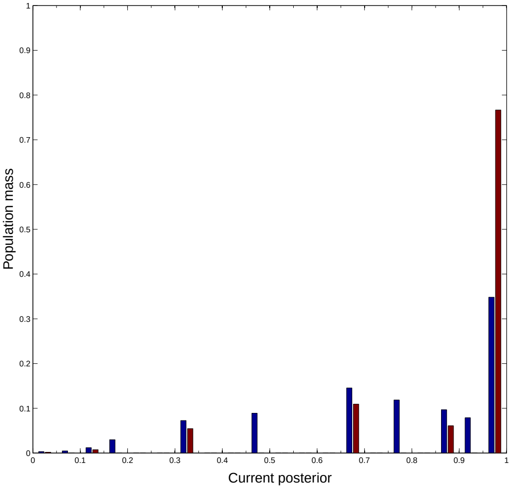

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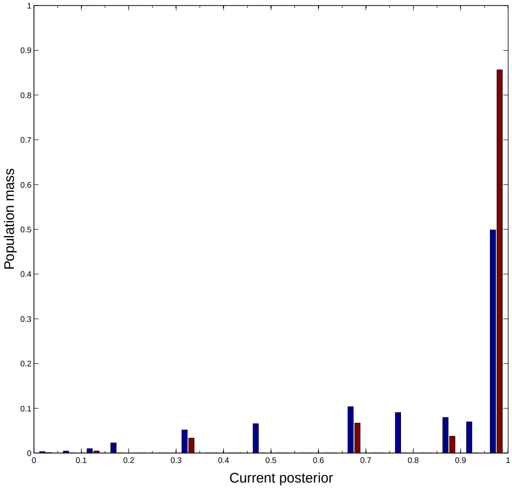

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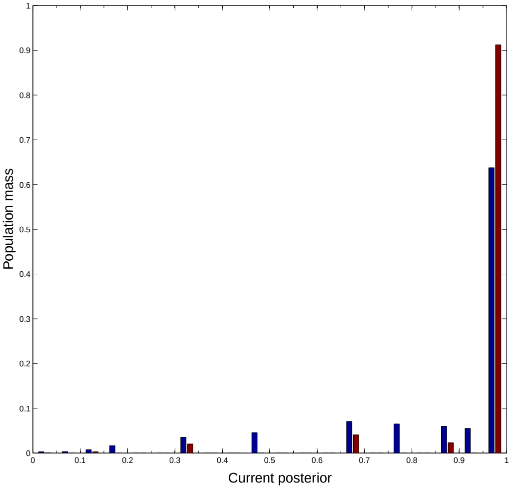

# Equilibrium Search Dynamics

With Manso and Malamud

- Signals and $X$ are joint Gaussian, with $\text{corr}(X, s_i) = \rho$.
- Agents arrive at a rate proportional to the population size, and leave at exponentially distributed times, pairwise independently.
- Agents meet others at a mean rate proportional to the rate at which they choose to expend search costs.
- At entry, agent $i$ receives $N_{i0}$ signals, *iid* across agents.
- At exit, an agent chooses an action $A$, with cost $(X - A)^2$.
- The optimal exit action is $A = E(X \mid \mathcal{F}_{it})$, so the expected exit cost is the $\mathcal{F}_{it}$-conditional variance of $X$,

$$\sigma_{it}^2 = v(N_{it}) \equiv \frac{1 - \rho^2}{1 + \rho^2(N_{it} - 1)}.$$

# Information Transmission

- Agent $i$ has current mean $E(X \mid \mathcal{F}_{it})$ and "precision" $N_{it}$.
- When agents $i$ and $j$ meet, their posterior precisions become $N_{it} + N_{jt}$. Their posterior means are given by the usual precision-weighted average of their priors.
- At constant meeting intensity $\lambda$, the cross-sectional precision distribution $\mu_t$ thus behaves as before, except for the effect of arrivals and departures.

# Search Technology

- Random matching (Stosszahlanzatz).
- Given current effort $c$ by an agent, the mean rate of matching with someone from a unit mass of agents using search effort $b$ is $cb$.
- The rate of cost of search effort $c$ is $K(c)$, for $c \in [c_L, c_H]$.

# Separability of Posterior Precision and Mean

*Proposition.* For any search-effort policy function $C : \mathbb{N} \rightarrow [c_L, c_H]$, the cross-sectional density $f_t$ of precisions and posterior means of the agents is almost surely given by

$$f_t(n, y, \omega) = \mu_t(n)\, p_n(y \mid X(\omega)), \qquad (3)$$

where $\mu_t$ is the unique solution of the Boltzmann equation for the evolution of the cross-sectional distribution of information precision and $p_n(\,\cdot \mid X)$ is the $X$-conditional Gaussian density of $E(X \mid s_1, \ldots, s_n)$, for any $n$ signals $s_1, \ldots, s_n$.

# Stationary Measure

In a stationary setting with search policy $C$, the precision distribution $\mu$ solves

$$0 = \eta(\gamma - \mu) + \mu^C * \mu^C - \mu^C \, \mu^C(\mathbb{N}),$$

where

- $\eta$ is the mean replacement rate of agents.
- $\gamma$ is the distribution of $N_{i0}$.
- $\mu^C$ is the effort-weighted measure, with $\mu^C(n) = C(n)\mu(n)$.

# Stationary Measure

*Lemma.* Given any policy $C$, there is a unique measure $\mu$ satisfying the stationary-measure equation.

This measure $\mu$ is characterized in the paper, and under technical conditions is the pointwise limit of $\mu_t$, invariant to $\mu_0$.

# Optimality and Equilibrium

Given a policy $C$ for other agents, agent $i$ has the value function $V(\cdot)$ defined by

$$V(N_{it}) = \operatorname{ess}\,\sup_{\phi}\; E\left(-e^{-r(\tau-t)}v(N_{i\tau}) - \int_0^{\tau} e^{-r(u-t)}K(\phi_u)\,du \;\middle|\; \mathcal{F}_{it}\right),$$

which is characterized by the associated HJB equation.

The policy $C$ is an equilibrium if $\phi_t^* = C(N_{it})$ is optimal.

# Trigger Policies

A trigger policy $C^N$, for some integer $N \geq 1$, is defined by

$$\begin{aligned} C_n^N &= c_H, \quad n < N, \\ &= c_L, \quad n \geq N. \end{aligned}$$

# Information Sharing Opportunities

*Proposition.* Let $\mu^M$ and $\nu^N$ be the unique stationary measures corresponding to trigger policies $C^M$ and $C^N$ respectively. Let $\mu^{C,N}(n) = \mu^N(n)C^N(n)$ denote the associated search-effort-weighted measure. If $N > M$, then $\mu^{C,N}$ has first order stochastic dominance (FOSD) over $\mu^{C,M}$.

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This comparison result need not hold for non-trigger policies!

There exist cases with $B \leq C$ but $\mu^B$ having FOSD over $\mu^C$.

# Optimal Effort is Decreasing in Precision

*Proposition.* Suppose that $K$ is increasing, convex, and differentiable. Then, given any population behavior $(\mu, C)$, the optimal search effort policy function is decreasing in precision.

*Proposition.* Suppose that $K(c) = \kappa c$ for some scalar $\kappa > 0$. Then, given $(\mu, C)$, some a trigger policy $C^N$ that is optimal for all agents.

# Existence of Equilibrium

*Theorem.* Suppose that $K(c) = \kappa c$ for some scalar $\kappa > 0$. Then there exists a trigger policy that is an equilibrium.

# The Equilibrium Impact of a Search Subsidy

- A tax $\tau$ is charged to each agent entering the market
- The proceeds are used to subsidize search so that the search cost is $K_\delta(c) = (\kappa - \delta)c$.

*Proposition.* If $C^N$ is an equilibrium with subsidy $\delta$, then for any $\delta' \geq \delta$, there exists some $N' \geq N$ such that $C^{N'}$ is an equilibrium with subsidy $\delta'$.

# Example

1. For some integer $N > 1$, $\gamma(0) = 1/2$, $\gamma(N) = 1/2$, and $c_L = 0$.
2. It is possible to choose parameters so that, given market conditions $(\mu^N, C^N)$, agents slightly prefer policy $C^0$ over $C^N$.
3. We can choose the subsidy rate $\delta$ so that, given market conditions $(\mu^N, C^N)$, agents strictly prefer $C^N$ to $C^0$.
4. For sufficiently large $N$ all agents have strictly higher indirect utility.

# The Equilibrium Impact of Public Information

Agents are given $M \geq 1$ additional public signals at entry.

**Proposition** If $C^N$ is an equilibrium with $M$ public signals, then for any $M' \leq M$, there exists some $N' \geq N$ such that $C^{N'}$ is an equilibrium with $M'$ public signals.

We provide examples with strict dominance.

# Example

1. Suppose, for some integer $N > 1$, that $\pi_0 = 1/2$, $\pi_N = 1/2$, and $c_L = 0$.
2. Choose parameters so that, given market conditions $(\mu^N, C^N)$ agents are indifferent between policies $C^N$ and $C^0$.
3. Give each agent $M = 1$ public signal at entry.
4. All agents strictly prefer $C^0$ to $C^N$
5. For sufficiently large $N$ all agents have strictly lower indirect utility.

# Conclusion

- Model of social learning with endogenous search intensity.
- Social learning may slow down or even collapse:
  - coordination problems.
  - externality problems.
- Two policy interventions:
  - search subsidy.
  - education at entry.

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