Optimal Finite Horizon Control of Manufacturing Systems: Fluid Solution by SCLP (separated continuous LP) and Fluid Tracking using IVQs (infinite virtual queues)

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Control of computer chip semi-conductor wafer fab

Cost: 3×10^9 \$
Return: 3 years
Cycle time: 6 weeks
WIP: 60,000 wafers, 180×10^6 \$
Challenge: Control the queues at ~500 work steps

No steady state,
Finite horizon control,
Rolling horizon update

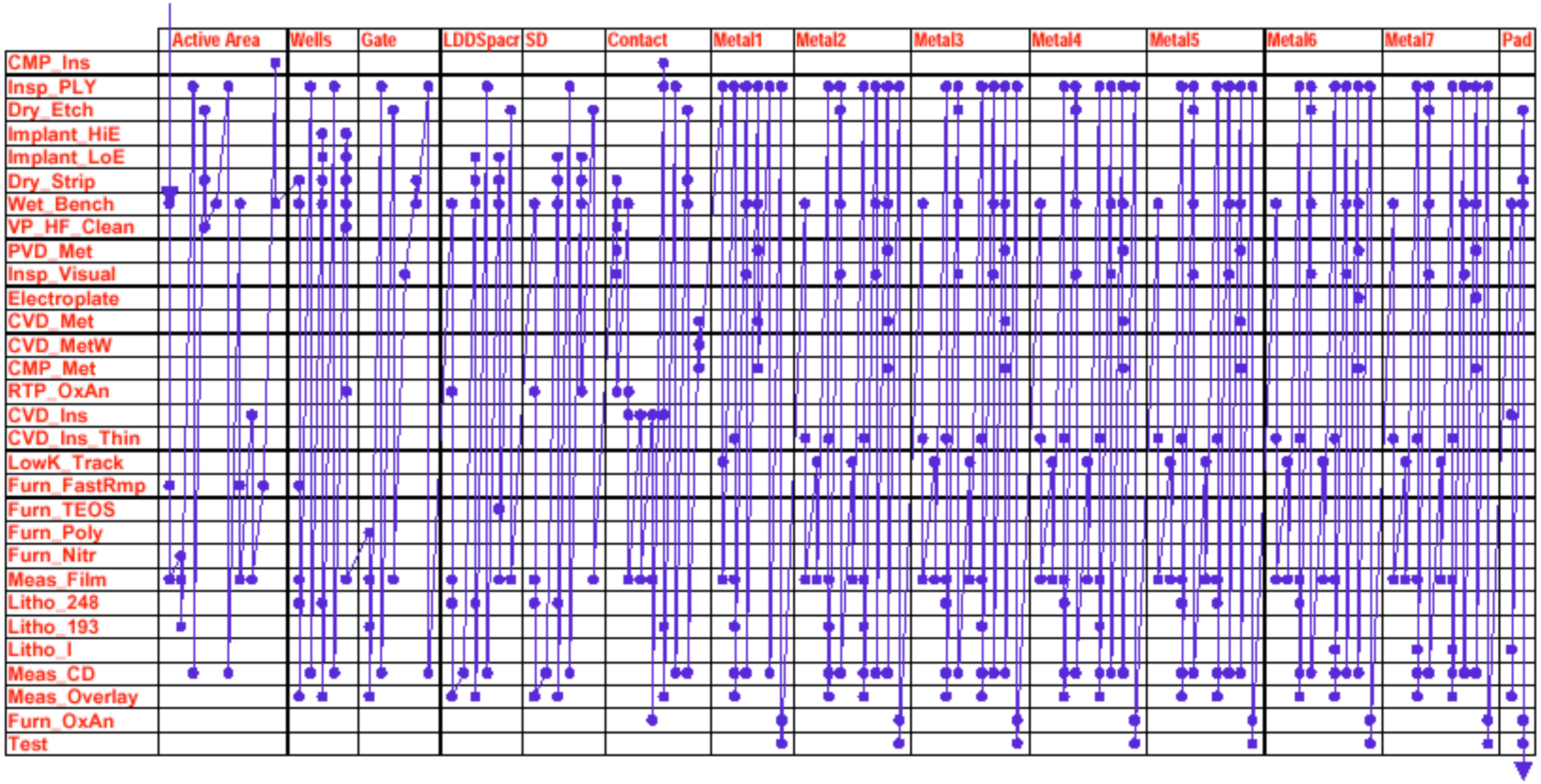


Figure 1. Generic 130-nm Semiconductor Process Flow Showing Massively Reentrant Product Flows

Finite horizon control of multi class queueing networks MCQN

Control MCQN $(Q(t),T(t))$, over $0<t<T$

$$Q_k(t)=Q_k(0)-S_k(T_k(t))+\sum_{k'}\Phi_{k'k}(S_{k'}(T_{k'}(t))) \quad k\in K$$

objective $$\min\sum_k \gamma_k T_k(T)+c_k\int_0^T Q_k(t)dt$$

Fluid Problem

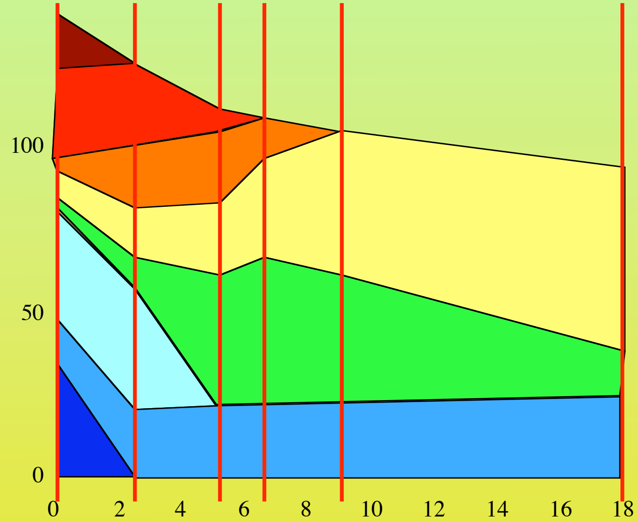
$$V^*=\min\int_0^T\big(\gamma' u(s)+c' q(s)\big)ds$$

$$s.t. \int_0^t Ru(s)ds+q(t)=q(0),$$

$$Au(t)\le 1,$$

$$u(t),q(t)\ge 0, \quad t\in[0,T]$$

Optimal Fluid Solution



Tracking of Fluid:

- Model Deviations from Fluid as MCQN with IVQ
- Use Max Pressure to keep Deviations Stable

Asymptotically Optimal

Continuous linear programming

Bellman (1953) **Economic models,** Dantzig, Perold, Anstreicher.

$$\max \int_0^T \underline{c}'(s)\underline{u}(s)\,ds$$

$$H(t)\,\underline{u}(t) + \int_0^t G(s,t)\underline{u}(s)\,ds \le \underline{a}(t) \qquad CLP$$

$$\underline{u}(t) \ge 0,\ 0 < t < T$$

Anderson (1975) **job shop scheduling,**

$$\max \int_0^T c(T-s)'\,u(s)\,ds$$

$$\int_0^t G\,u(s)\,ds + x(t) = a(t) \qquad SCLP$$

$$H\,u(t) \le b(t)$$

$$x,\ u \ge 0, \qquad 0 < t < T$$

Anderson, Philpott & Nash papers and **Book** (1987),.

Pullan (1993) **Dual problem, Strong duality, convergent (not finite) algorithm**

SCLP for MCQN

$$\max \int_0^T \left(\left(\gamma' + (T-t)c' \right) u(t) + d'x(t) \right) dt$$

$$s.t. \int_0^t G \;\; u(s)ds \qquad + Fx(t) \le \alpha + at$$

$$H \;\; u(t) \qquad\qquad \le b$$

$$u(t), x(t) \ge 0 \qquad 0 < t < T$$

Variables: $K+J+I+L$

$G \;\; K\text{x}J \qquad F \;\; K\text{x}L$

$H \;\; I\text{x}J$

States x_k: $k=1,\dots,K$, $k=K+1,\dots,K+L$ (slacks),

Controls u_j: $j=1,\dots,J$, $j=J+1,\dots,J+I$ (slacks)

Structure:

Time horizon: $\boxed{T}$

Control cost: $\boxed{\gamma \quad 0 \quad 0 \quad 0}$

Inventory cost: $\boxed{c \quad 0 \quad 0 \quad d}$

System:

$$\begin{bmatrix} G & 0 & I & F \\ H & I & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} a \\ b \end{bmatrix} \begin{matrix} \text{input rate} \\ \text{control capacity} \end{matrix} \quad \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \begin{matrix} \text{initial state} \\ \; \end{matrix}$$

Solving SCLP Discretization vs Simplex

Discrete Time: CLP and SCLP become LP

(1) LP is large

(2) approximation is doubtful

(3) problem structure is lost

Most past approaches (including Pullan) used discrete approximation

SCLP provides excellent models, but because it needed discretization,
problems were modeled as dictrete time multi-period LPs

We solve SCLP in continuous time, exactly, in a finite number of steps.
Our algorithm performs simplex steps, in function space.
We gain much insight to the solution, e.g. sensitivity analysis in time and space

A simplex approach

Solution features: partition of time horizon $0=t_0<t_1<\ldots<t_N=T$
piecewise constant controls $u(t)$
continuous piecewise linear $x(t)$

- Symmetric dual
- Extreme points are sequences of bases
- Edges via validity regions
- SCLP Pivots
- Parametric sequence of steps

$$\max \int_0^T \left((\gamma' + (T-t)c')u(t) + d'x(t) \right) dt$$

$$s.t. \int_0^t G\, u(s)ds + Fx(t) \le \alpha + at$$

$$H\, u(t) \le b$$

$$u(t), x(t) \ge 0 \qquad 0 < t < T$$

Simplex in the space of bounded measurable (density) controls

Essential Assumption
Problem Non-Degenerate:

$\begin{bmatrix} a \\ b \end{bmatrix}$ is in general position to $\begin{bmatrix} G & 0 & I & F \\ H & I & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} c \\ d \end{bmatrix}$ is in general position to $\begin{bmatrix} G' & 0 & -I & H' \\ F' & -I & 0 & 0 \end{bmatrix}$

Symmetric Duality & Complementary Slackness

$$\max \int_0^T \left((\gamma' + (T-t)c')u(t) + d'x(t) \right) dt$$

$$\begin{aligned} s.t. \int_0^t G\ u(s)ds \quad & + F x(t) \le \alpha + at \\ H\ u(t) \quad & \le b \\ u(t), x(t) \ge 0 \quad & 0 < t < T \end{aligned}$$

$$\int_0^T x_k(t) p_k(T-t)dt = \int_0^T q_j(T-t)u_j(t)dt = 0$$

primal states $x_k(t)$ — primal controls $u_j(t)$

dual states $q_j(T-t)$ — dual controls $p_k(T-t)$

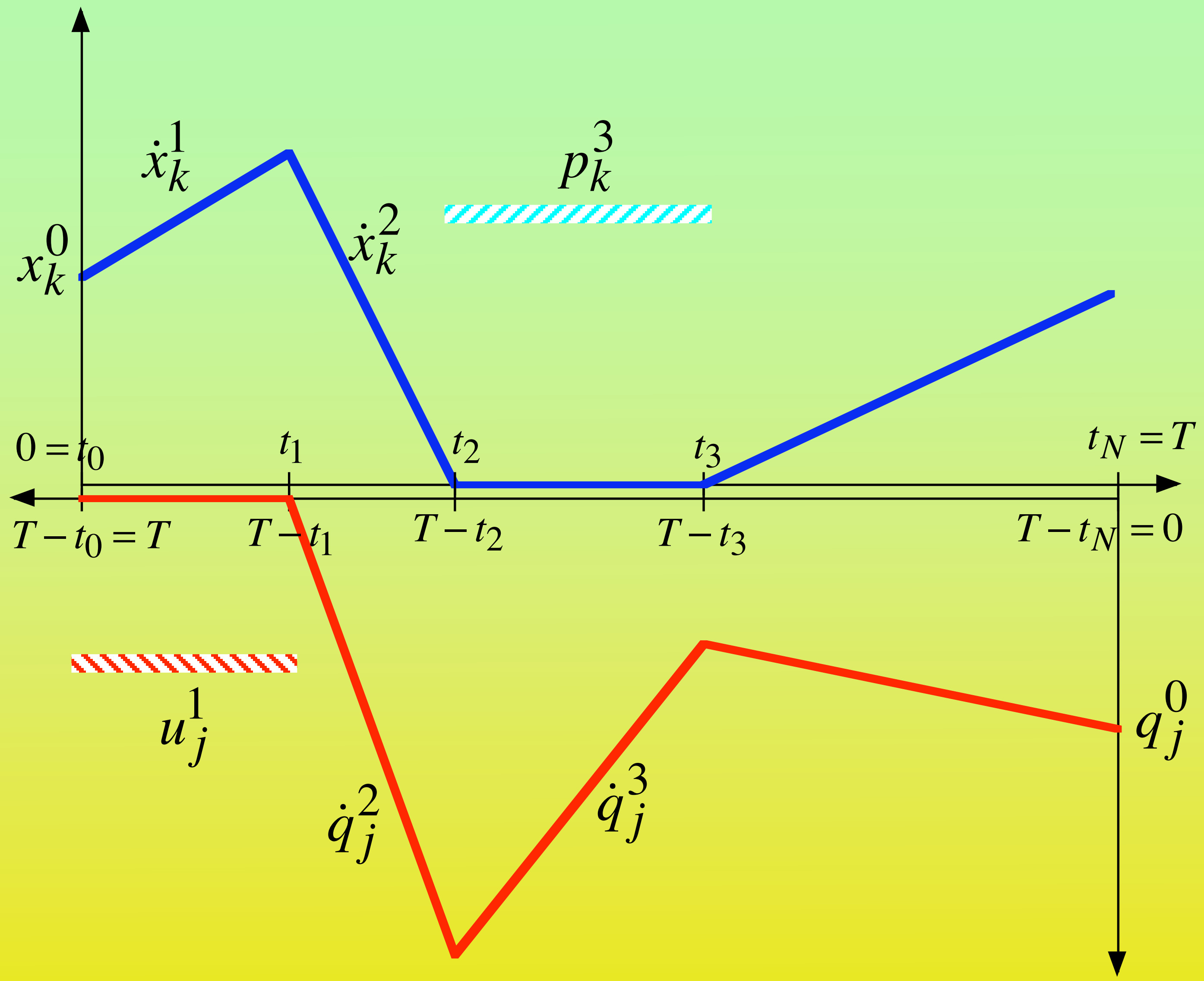
$$\min \int_0^T \left((\alpha' + (T-t)a')p(t) + b'q(t) \right) dt$$

$$\begin{aligned} s.t. \int_0^t G'\ p(s)ds \quad & + H'q(t) \ge \gamma + ct \\ F'\ p(t) \quad & \ge d \\ p(t), q(t) \ge 0 \quad & 0 < t < T \end{aligned}$$

Dual runs in reversed time

We produce primal dual feasible complementary slack solutions

Complementary slack optimal solutions



Boundary and Rates LP/LP*

Stage 1

Solve for $T = 0$

(horizon dT)

Determine $x(0)$, $q(0)$

$$\boxed{T}$$

γ	0	0	0		
c	0	0	d		
G	0	I	F	a	α
H	I	0	0	b	0

$$\max \int_0^T \left((\gamma' + (T-t)c')u(t) + d'x(t) \right) dt$$

$$s.t. \int_0^t G\, u(s)ds \quad + Fx(t) \le \alpha + at$$

$$H\, u(t) \quad \le b$$

$$u(t), x(t) \ge 0 \qquad 0 < t < T$$

Boundary LP

$$\max \quad d'x^0 \qquad\qquad \min \quad b'q^N$$

$$s.t.\ Fx^0 \le \alpha,\ x^0 \ge 0 \qquad\qquad s.t.\ H'q^N \ge \gamma,\ q^N \ge 0$$

$x(0)$, $q(0)$ remain unchanged for all time horizons

Boundary and Rates LP/LP*

Boundary LP

$$\max \quad d'x^0 \qquad\qquad \min \quad b'q^N$$

$$s.t.\ Fx^0 \le \alpha, x^0 \ge 0 \qquad s.t.\ H'q^N \ge \gamma,\ q^N \ge 0$$

Determine $x(0)$, $q(0)$

Stage 2
Determine
values of $u(t), p(T-t)$
slopes of $x(t), q(T-t)$
for $t_{n-1}<t<t_n$

$$\boxed{T}$$

$$\boxed{\gamma \quad 0 \quad 0 \quad 0}$$

$$\boxed{c \quad 0 \quad 0 \quad d}$$

$$\boxed{\begin{matrix} G & 0 & I & F \\ H & I & 0 & 0 \end{matrix}} \quad \boxed{\begin{matrix} a \\ b \end{matrix}} \quad \boxed{\begin{matrix} \alpha \\ 0 \end{matrix}}$$

$$\max \int_0^T \left((\gamma' + (T-t)c')u(t) + d'x(t) \right) dt$$

$$s.t. \int_0^t G\ u(s)ds \quad + Fx(t) \le \alpha + at$$

$$H\ u(t) \quad \le b$$

$$u(t), x(t) \ge 0 \qquad 0 < t < T$$

Rates LP

$$\max \ c'u^n + d'\dot{x}^n \qquad\qquad \min \ a'p^n + b'\dot{q}^n$$

$$s.t. \quad Gu^n + F\dot{x}^n \le a \qquad s.t. \quad G'p^n + H'\dot{q}^n \ge c$$

$$Hu^n \quad \le b \qquad\qquad F'p^n \quad \ge d$$

Boundary and Rates LP/LP*

Boundary LP

$$\max \quad d'x^0 \qquad\qquad \min \quad b'q^N$$

$$s.t.\ Fx^0 \le \alpha, x^0 \ge 0 \qquad\qquad s.t.\ H'q^N \ge \gamma,\ q^N \ge 0$$

Determine $x(0), q(0)$

Rates LP

$$\begin{aligned} \max\ & c'u^n + d'\dot{x}^n \\ s.t.\ & Gu^n + F\dot{x}^n \le a \\ & Hu^n \le b \end{aligned} \qquad\qquad \begin{aligned} \min\ & a'p^n + b'\dot{q}^n \\ s.t.\ & G'p^n + H'\dot{q}^n \ge c \\ & F'p^n \ge d \end{aligned}$$

Determine
values of $u(t), p(T\text{-}t)$
slopes of $x(t), q(T\text{-}t)$
for $t_{n-1}<t<t_n$

Sign constraints for the interval $t_{n-1}<t<t_n$

$$\textit{If} \quad x_k(t_{n-1}) > 0 \quad \textit{then} \quad \dot{x}_k^n \text{ unrestricted}, \quad p_k^n = 0$$

$$\textit{If} \quad q_j(T - t_n) > 0 \quad \textit{then} \quad \dot{q}_j^n \text{ unrestricted}, \quad u_j^n = 0$$

$$\textit{Else}: \quad \dot{x}_k^n, u_j^n, \dot{q}_j^n, p_k^n \ge 0$$

Solution Structure theorem

SCLP solution is determined by partition of time horizon $0=t_0<t_1<...<t_N=T$ and a sequence of bases $B_1, \ldots, B_N$

$$B_n : \ \dot{x}^n, u^n, \dot{q}^n, p^n \qquad B_n \to B_{n+1} \text{ adjacent} \qquad \tau_n = t_n - t_{n-1}$$

$$B_n \to B_{n+1} : \begin{cases} \dot{x}_k \ \text{ leaves the basis} \Rightarrow 0 = \ x_k(t_n) \ = x_k^0 + \sum_{m=1}^{n} \tau_m \dot{x}_k^m \\ u_j \ \text{ leaves the basis} \Rightarrow 0 = q_j(T-t_n) = q_j^N + \sum_{m=n+1}^{N} \tau_m \dot{q}_j^m \end{cases}$$

The Structure Theorem

Equations and inequalities
for interval lengths and slacks

$$\begin{bmatrix} 1\cdots1 & 0 \\ A & 0 \\ B & -I \end{bmatrix} \begin{bmatrix} \tau \\ \sigma \end{bmatrix} = \begin{bmatrix} T \\ g \\ h \end{bmatrix}$$

Theorem If a base-sequence $B_1,\ldots,B_N$ is:
Admissible (u,p≥0), adjacent, consistent with the boundary (x^0,q^N)
and has interval lengths and slacks $\tau,\sigma > 0$,
then the solution u,x,p,q is optimal

Conversely: almost all solutions are of this form

These Base-sequences are the extreme points

Validity Regions

Theorem An optimal base-sequence $B_1,\ldots,B_N$ is optimal for a **convex polyhedral cone** of boundary values x^0, q^N, T .

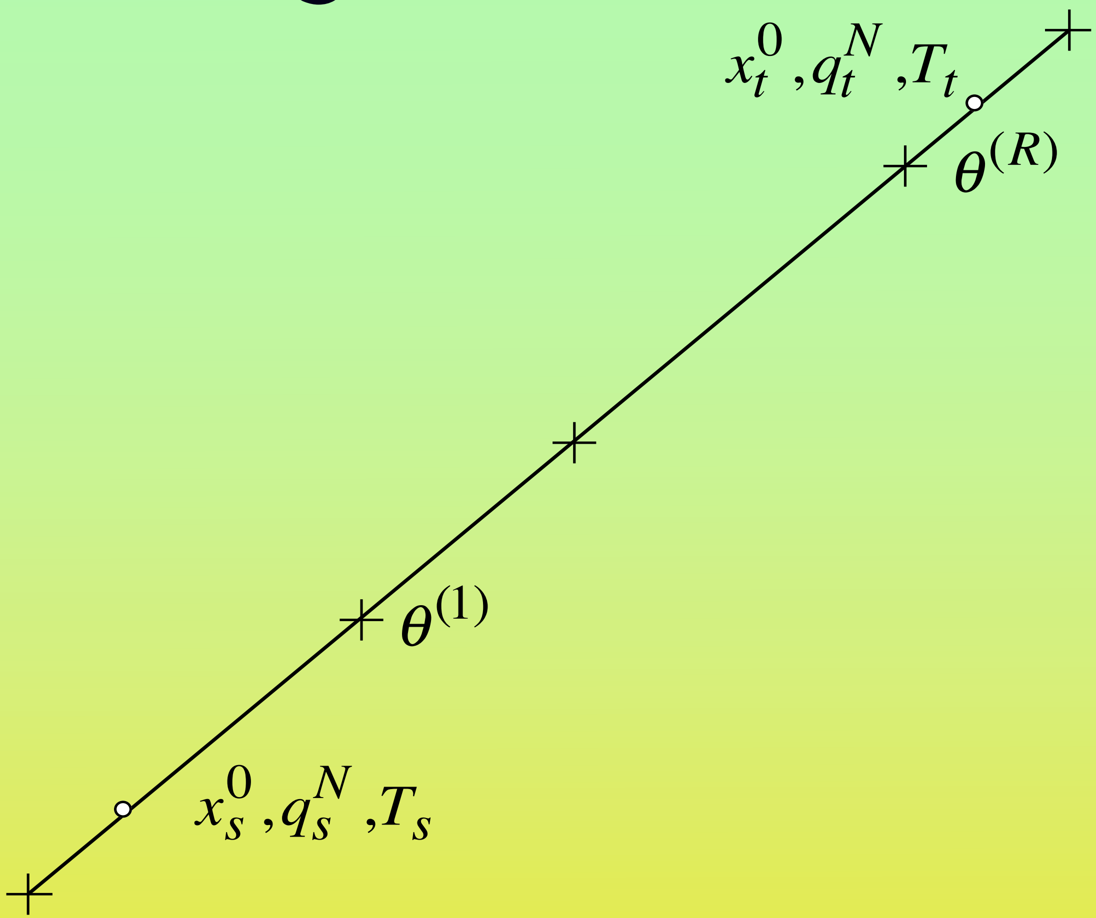
We call this the **Validity Region** of the base sequence

Definition: Two base sequences whose validity regions touch are **Neighboring** base sequences.

This defines an **Edge**

Moving along an edge is an **SCLP Pivot**.

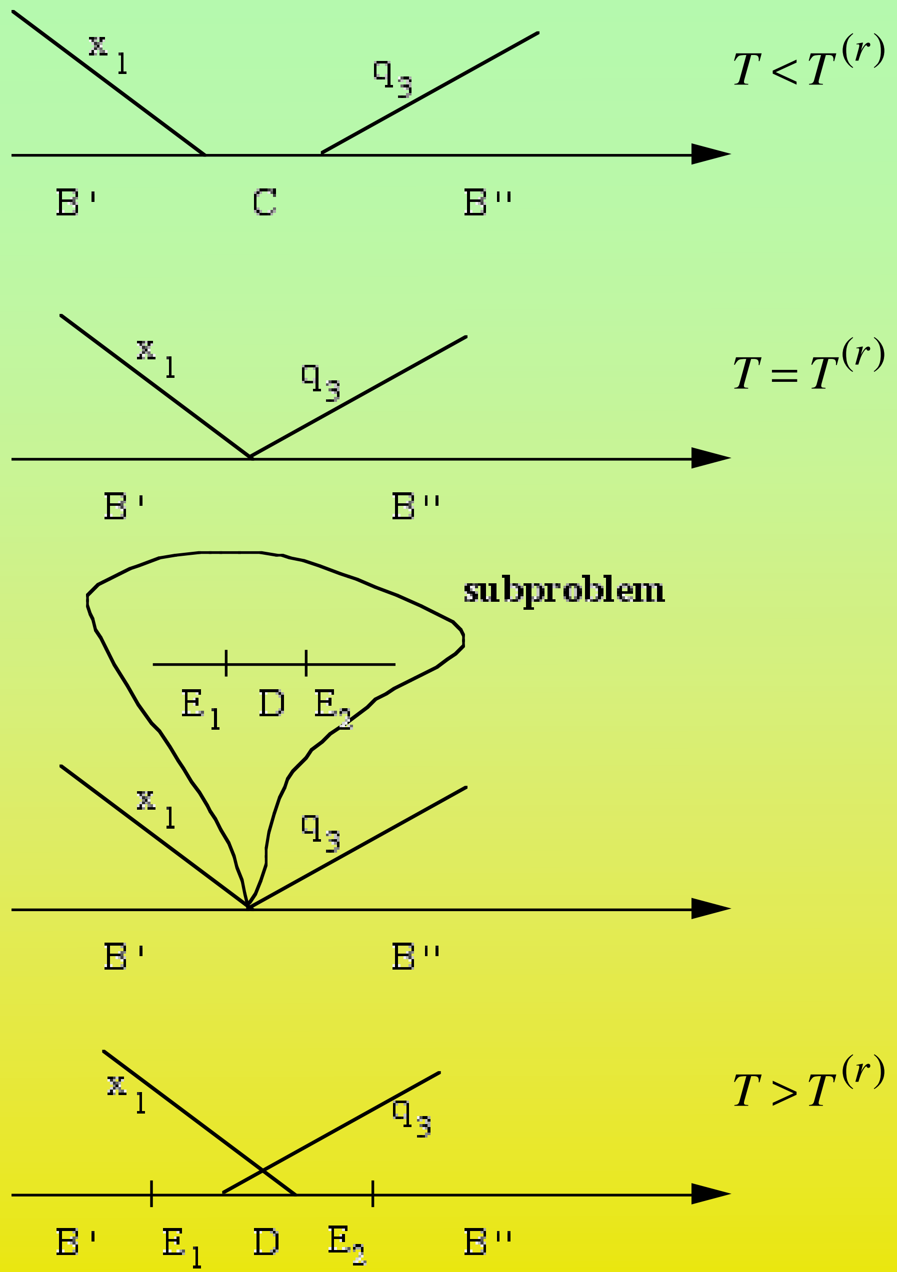
Parametric Algorithm



Algorithm pivots at $\underline{\theta} \le 0 < \theta^{(1)} < \cdots < \theta^{(R)} < 1 \le \overline{\theta}$

Moving from x_s^0, q_s^N, T_s to x_t^0, q_t^N, T_t

SCLP pivot



A necessary condition for pivoting is that the problem is completely non-degenerate:

$\begin{bmatrix} a \\ b \end{bmatrix}$ is in general position to $\begin{bmatrix} G & 0 & I & F \\ H & I & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} c \\ d \end{bmatrix}$ is in general position to $\begin{bmatrix} G' & 0 & -I & H' \\ F' & -I & 0 & 0 \end{bmatrix}$

If that condition does not hold we have no way to find where to `repair' solution after $T^{(r)}$

Simple SCLP example

$$\max \int_0^6 \left((8-t)u_1(t) - x_2(t) \right) dt$$

$$\begin{aligned} s.t. \quad & \int_0^t u_1(s)ds && + x_2(t) \le 3+t \\ & u_1(t) && \le 2 \\ & u, x \ge 0 \end{aligned}$$

Simple SCLP example 1

$$\max \int_0^T \big((-4+2(T-t))u_1(t) - x_2(t)\big)\, dt$$

$$\begin{aligned} s.t. \quad & \int_0^t u_1(s)ds && + x_2(t) \le 3+t \\ & u_1(t) && \le 2 \\ & u, x \ge 0 \end{aligned}$$

Simple SCLP example 2

$$\max \int_0^T \left((-4 + 2(T-t)) u_1(t) - x_2(t) \right) dt$$

$$\text{s.t.} \quad \int_0^t u_1(s) ds \quad + x_2(t) \le 3 + t$$

$$u_1(t) \quad \le 2$$

$$u, x \ge 0$$

$$\min \int_0^T \left((3 + T - t) p_1(t) \quad + 2 q_2(t) \right) dt$$

$$\text{s.t.} \quad \int_0^t p_1(s) ds \quad + q_2(t) \ge -4 + 2t$$

$$p_1(t) \quad \ge -1$$

$$p, q \ge 0$$

Simple SCLP example 3

$$\max \int_0^T \left((-4+2(T-t))u_1(t) - x_2(t) \right) dt$$

$$\text{s.t.} \quad \int_0^t u_1(s)ds \quad + x_1(t) + x_2(t) = 3+t$$

$$u_1(t) \; + u_2(t) \qquad = 2$$

$$u, x \geq 0$$

$$\min \int_0^T \left((3+T-t)p_1(t) \quad + 2q_2(t) \right) dt$$

$$\text{s.t.} \quad \int_0^t p_1(s)ds \quad - q_1(t) + q_2(t) = -4+2t$$

$$p_1(t) \; - p_2(t) \qquad = -1$$

$$p, q \geq 0$$

Simple SCLP example 4

$$\max \int_0^T \left((-4+2(T-t))u_1(t) - x_2(t) \right) dt$$

$$s.t. \quad \int_0^t u_1(s)ds \quad + x_1(t) + x_2(t) = 3+t$$

$$u_1(t) + u_2(t) = 2$$

$$u, x \geq 0$$

$$\min \int_0^T \left((3+T-t)p_1(t) \quad + 2q_2(t) \right) dt$$

$$s.t. \quad \int_0^t p_1(s)ds \quad - q_1(t) + q_2(t) = -4 + 2t$$

$$p_1(t) - p_2(t) = -1$$

$$p, q \geq 0$$

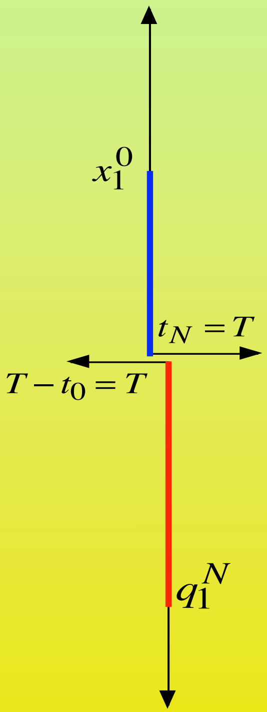
Boundary LP $T=0$

$$\max \; -x_2^0$$

$$s.t. \quad x_1^0 + x_2^0 = 3$$

$$x^0 \geq 0$$

Solution $x_1^0 = 3$



$$\min \quad 2q_2^N$$

$$s.t. \quad -q_1^N + q_2^N = -4$$

$$q^N \geq 0$$

Solution $q_1^N = 4$

Simple SCLP example 7

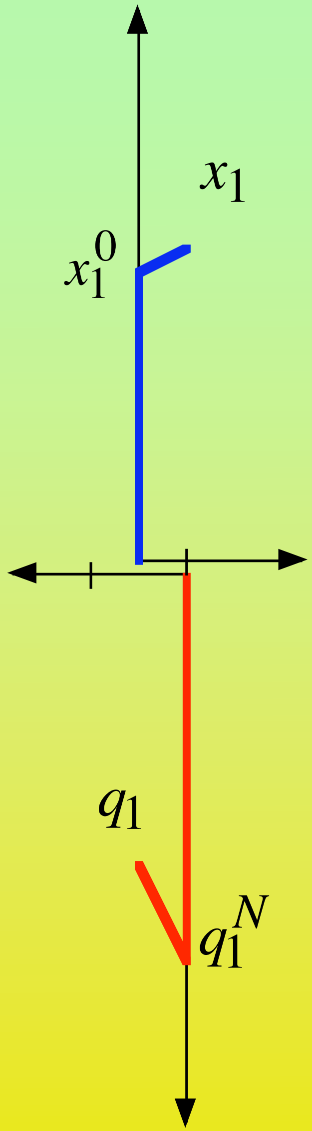
Boundary LP

$$\max \; -x_2^0$$
$$s.t. \quad x_1^0 + x_2^0 = 3$$
$$x^0 \geq 0$$

Solution $x_1^0 = 3$

$$\min \quad 2q_2^N$$
$$s.t. \quad -q_1^N + q_2^N = -4$$
$$q^N \geq 0$$

Solution $q_1^N = 4$



Rates LP $0<T<2$

$$\max \; 2u_1 \qquad - \dot{x}_2$$
$$s.t. \quad u_1 + \dot{x}_1 + \dot{x}_2 = 1$$
$$u_1 + u_2 \qquad = 2$$

$\dot{x}_1$ **U**, u_1 **Z**, $\dot{x}_2, u_2$ **P**

Solution $u_2 = 2, \; \dot{x}_1 = 1$

$$\min \quad p_1 \qquad + 2\dot{q}_2$$
$$s.t. \quad p_1 \quad - \dot{q}_1 + \dot{q}_2 = 2$$
$$p_1 - p_2 \qquad = -1$$

$\dot{q}_1$ **U**, p_1 **Z**, $p_2, \dot{q}_2$ **P**

Solution $p_2 = 1, \; \dot{q}_1 = -2$

Simple SCLP example 7

Boundary LP

$$\max \; -x_2^0$$
$$s.t. \quad x_1^0 + x_2^0 = 3$$
$$x^0 \geq 0$$

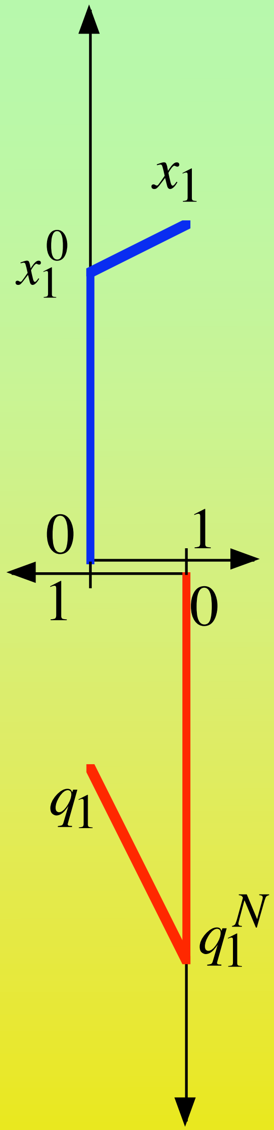
Solution $x_1^0 = 3$

Rates LP $0 < T < 2$

$$\max \; 2u_1 \qquad - \dot{x}_2$$
$$s.t. \quad u_1 \; + \dot{x}_1 + \dot{x}_2 = 1$$
$$u_1 + u_2 \qquad = 2$$

$\dot{x}_1 \; \mathbf{U}, u_1 \; \mathbf{Z}, \; \dot{x}_2, u_2 \; \mathbf{P}$

Solution $u_2 = 2, \; \dot{x}_1 = 1$



$$\min \qquad 2q_2^N$$
$$s.t. \quad -q_1^N + q_2^N = -4$$
$$q^N \geq 0$$

Solution $q_1^N = 4$

$$\min \quad p_1 \qquad + 2\dot{q}_2$$
$$s.t. \quad p_1 \quad - \dot{q}_1 + \dot{q}_2 = 2$$
$$p_1 - p_2 \qquad = -1$$

$\dot{q}_1 \; \mathbf{U}, \; p_1 \; \mathbf{Z}, \; p_2, \dot{q}_2 \; \mathbf{P}$

Solution $p_2 = 1, \; \dot{q}_1 = -2$

Simple SCLP example 7

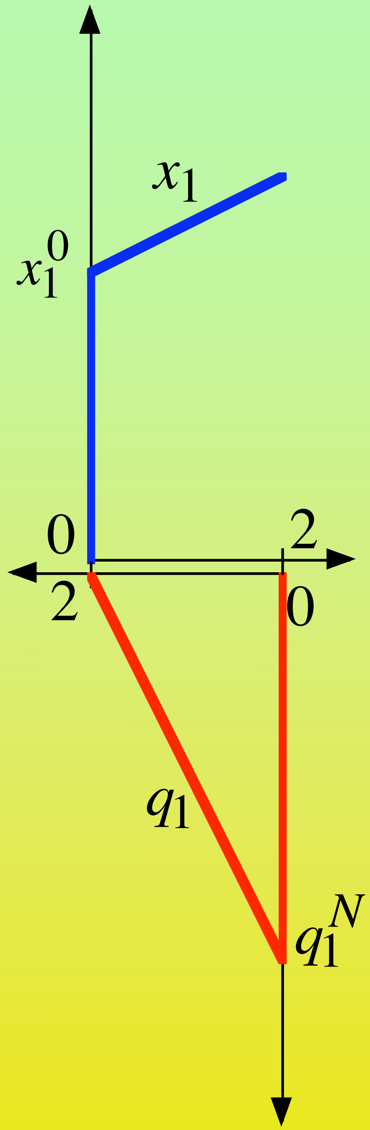
Boundary LP

$$\max \; -x_2^0$$
$$s.t. \quad x_1^0 + x_2^0 = 3$$
$$x^0 \ge 0$$

Solution $x_1^0 = 3$

$$\min \quad 2q_2^N$$
$$s.t. \quad -q_1^N + q_2^N = -4$$
$$q^N \ge 0$$

Solution $q_1^N = 4$



Rates LP $0<T<2$

$$\max \; 2u_1 \qquad - \dot{x}_2$$
$$s.t. \quad u_1 \; + \dot{x}_1 + \dot{x}_2 = 1$$
$$u_1 + u_2 \qquad = 2$$

$\dot{x}_1$ **U**, u_1 **Z**, $\dot{x}_2, u_2$ **P**

Solution $u_2 = 2, \; \dot{x}_1 = 1$

$$\min \quad p_1 \qquad + 2\dot{q}_2$$
$$s.t. \quad p_1 \quad - \dot{q}_1 + \dot{q}_2 = 2$$
$$p_1 - p_2 \qquad = -1$$

$\dot{q}_1$ **U**, p_1 **Z**, $p_2, \dot{q}_2$ **P**

Solution $p_2 = 1, \; \dot{q}_1 = -2$

Simple SCLP example 8

Boundary LP

$$\max \; -x_2^0$$
$$s.t. \quad x_1^0 + x_2^0 = 3$$
$$x^0 \geq 0$$

Solution $x_1^0 = 3$

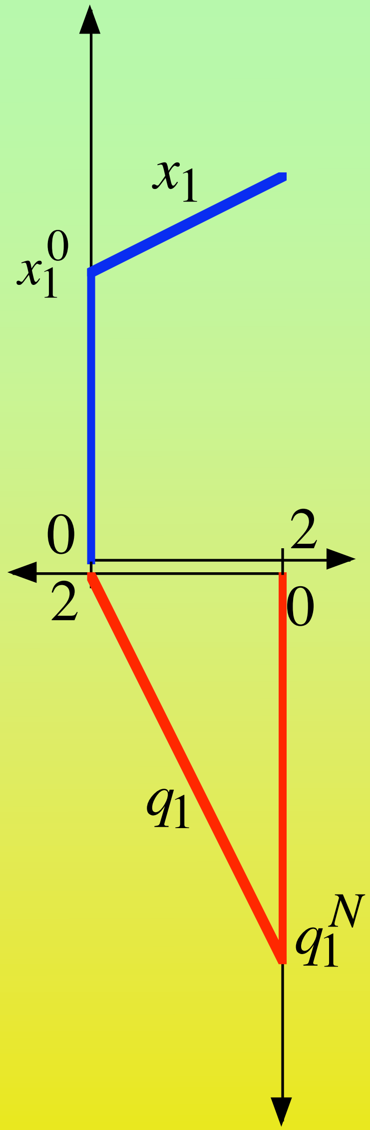
Rates LP $\;0<T<2$

$$\max \; 2u_1 \qquad - \dot{x}_2$$
$$s.t. \quad u_1 + \dot{x}_1 + \dot{x}_2 = 1$$
$$u_1 + u_2 \qquad = 2$$

$\dot{x}_1$ **U**, u_1 **Z**, $\dot{x}_2, u_2$ **P**

Solution $u_2 = 2, \; \dot{x}_1 = 1$

$\dot{x}_1$ **U**, $\dot{x}_2, u_1, u_2$ **P**



$$\min \qquad 2q_2^N$$
$$s.t. \quad -q_1^N + q_2^N = -4$$
$$q^N \geq 0$$

Solution $q_1^N = 4$

$$\min \quad p_1 \qquad + 2\dot{q}_2$$
$$s.t. \quad p_1 \quad - \dot{q}_1 + \dot{q}_2 = 2$$
$$p_1 - p_2 \qquad = -1$$

$\dot{q}_1$ **U**, p_1 **Z**, $p_2, \dot{q}_2$ **P**

Solution $p_2 = 1, \; \dot{q}_1 = -2$

p_1 **Z**, $\dot{q}_1, p_2, \dot{q}_2$ **P**

Simple SCLP example 8

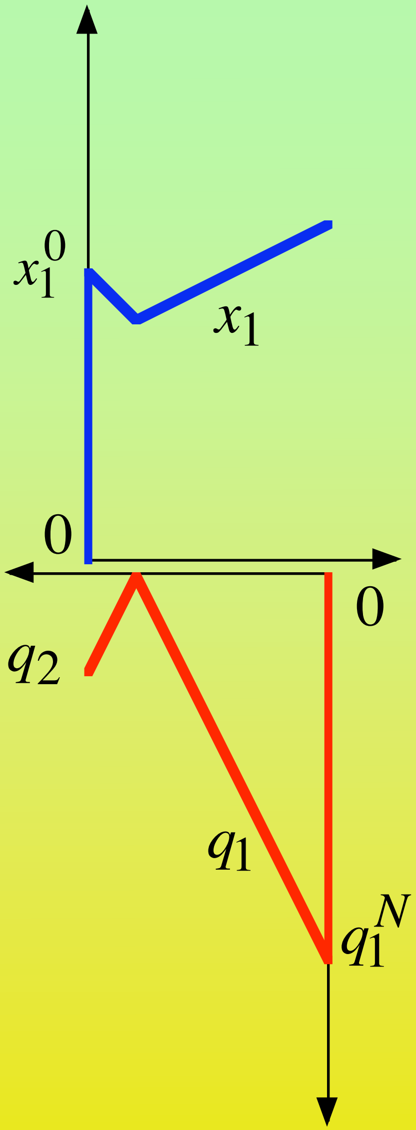
Boundary LP

$$\max \; -x_2^0$$
$$s.t. \quad x_1^0 + x_2^0 = 3$$
$$x^0 \geq 0$$

Solution $x_1^0 = 3$

$$\min \; 2q_2^N$$
$$s.t. \quad -q_1^N + q_2^N = -4$$
$$q^N \geq 0$$

Solution $q_1^N = 4$



Rates LP $2<T$

$$\max \; 2u_1 \quad - \dot{x}_2$$
$$s.t. \quad u_1 + \dot{x}_1 + \dot{x}_2 = 1$$
$$u_1 + u_2 = 2$$

$$\min \; p_1 \quad + 2\dot{q}_2$$
$$s.t. \quad p_1 - \dot{q}_1 + \dot{q}_2 = 2$$
$$p_1 - p_2 = -1$$

$\dot{x}_1$ **U**, u_1 **Z**, $\dot{x}_2, u_2$ **P**

Solution $u_2 = 2, \; \dot{x}_1 = 1$

$\dot{x}_1$ **U**, $\dot{x}_2, u_1, u_2$ **P**

Solution $u_1 = 2, \; \dot{x}_1 = -1$

$\dot{q}_1$ **U**, p_1 **Z**, $p_2, \dot{q}_2$ **P**

Solution $p_2 = 1, \; \dot{q}_1 = -2$

p_1 **Z**, $\dot{q}_1, p_2, \dot{q}_2$ **P**

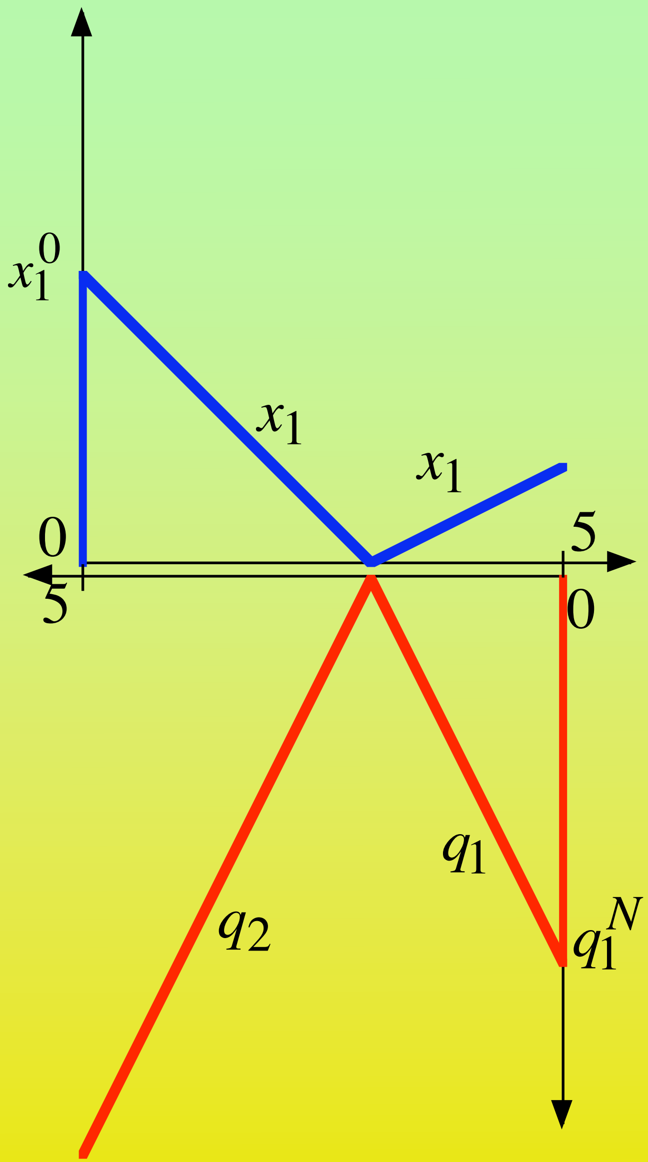
Solution $p_2 = 1, \; \dot{q}_2 = 2$

Simple SCLP example 9

Boundary LP

$$\max \; -x_2^0$$
$$s.t. \quad x_1^0 + x_2^0 = 3$$
$$x^0 \ge 0$$
Solution $x_1^0 = 3$

$$\min \; 2q_2^N$$
$$s.t. \quad -q_1^N + q_2^N = -4$$
$$q^N \ge 0$$
Solution $q_1^N = 4$



Rates LP $2<T<5$

$$\max \; 2u_1 \quad - \dot{x}_2$$
$$s.t. \quad u_1 + \dot{x}_1 + \dot{x}_2 = 1$$
$$u_1 + u_2 = 2$$

$$\min \; p_1 + 2\dot{q}_2$$
$$s.t. \quad p_1 - \dot{q}_1 + \dot{q}_2 = 2$$
$$p_1 - p_2 = -1$$

$\dot{x}_1$ **U**, u_1 **Z**, $\dot{x}_2, u_2$ **P**

Solution $u_2 = 2,\ \dot{x}_1 = 1$

$\dot{x}_1$ **U**, $\dot{x}_2, u_1, u_2$ **P**

Solution $u_1 = 2,\ \dot{x}_1 = -1$

$\dot{x}_1, \dot{x}_2, u_1, u_2$ **P**

$\dot{q}_1$ **U**, p_1 **Z**, $p_2, \dot{q}_2$ **P**

Solution $p_2 = 1,\ \dot{q}_1 = -2$

p_1 **Z**, $\dot{q}_1, p_2, \dot{q}_2$ **P**

Solution $p_2 = 1,\ \dot{q}_2 = 2$

$p_1, \dot{q}_1, p_2, \dot{q}_2$ **P**

Simple SCLP example 10

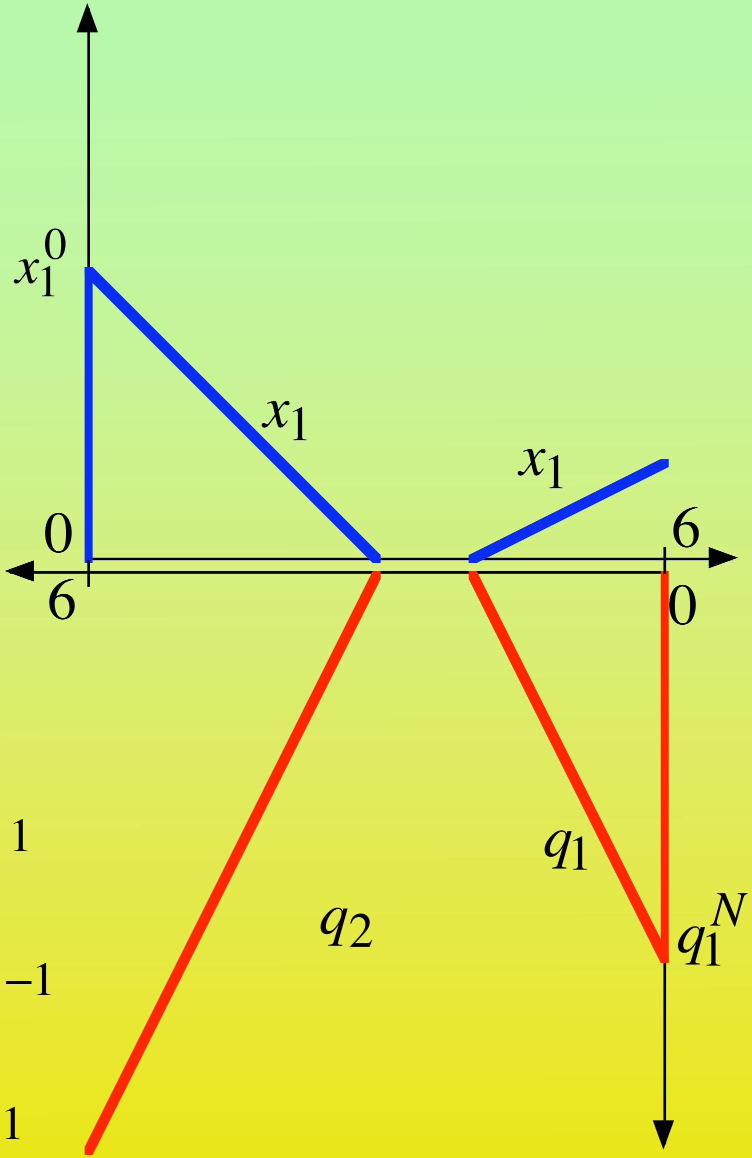
Boundary LP

$$\max \; -x_2^0$$
$$s.t. \quad x_1^0 + x_2^0 = 3$$
$$x^0 \ge 0$$

Solution $x_1^0 = 3$

$$\min \quad 2q_2^N$$
$$s.t. \quad -q_1^N + q_2^N = -4$$
$$q^N \ge 0$$

Solution $q_1^N = 4$



Rates LP $5 < T < \infty$

$$\max \; 2u_1 \quad - \dot{x}_2$$
$$s.t. \quad u_1 + \dot{x}_1 + \dot{x}_2 = 1$$
$$u_1 + u_2 = 2$$

$\dot{x}_1$ **U**, u_1 **Z**, $\dot{x}_2, u_2$ **P**

Solution $u_2 = 2, \; \dot{x}_1 = 1$

$\dot{x}_1$ **U**, $\dot{x}_2, u_1, u_2$ **P**

Solution $u_1 = 2, \; \dot{x}_1 = -1$

$\dot{x}_1, \dot{x}_2, u_1, u_2$ **P**

Solution $u_1 = 1, \; u_2 = 1$

$$\min \; p_1 \quad + 2\dot{q}_2$$
$$s.t. \quad p_1 \quad - \dot{q}_1 + \dot{q}_2 = 2$$
$$p_1 - p_2 = -1$$

$\dot{q}_1$ **U**, p_1 **Z**, $p_2, \dot{q}_2$ **P**

Solution $p_2 = 1, \; \dot{q}_1 = -2$

p_1 **Z**, $\dot{q}_1, p_2, \dot{q}_2$ **P**

Solution $p_2 = 1, \; \dot{q}_2 = 2$

$p_1, \dot{q}_1, p_2, \dot{q}_2$ **P**

Solution $p_1 = 2, \; p_2 = 3$

Simple SCLP example 11

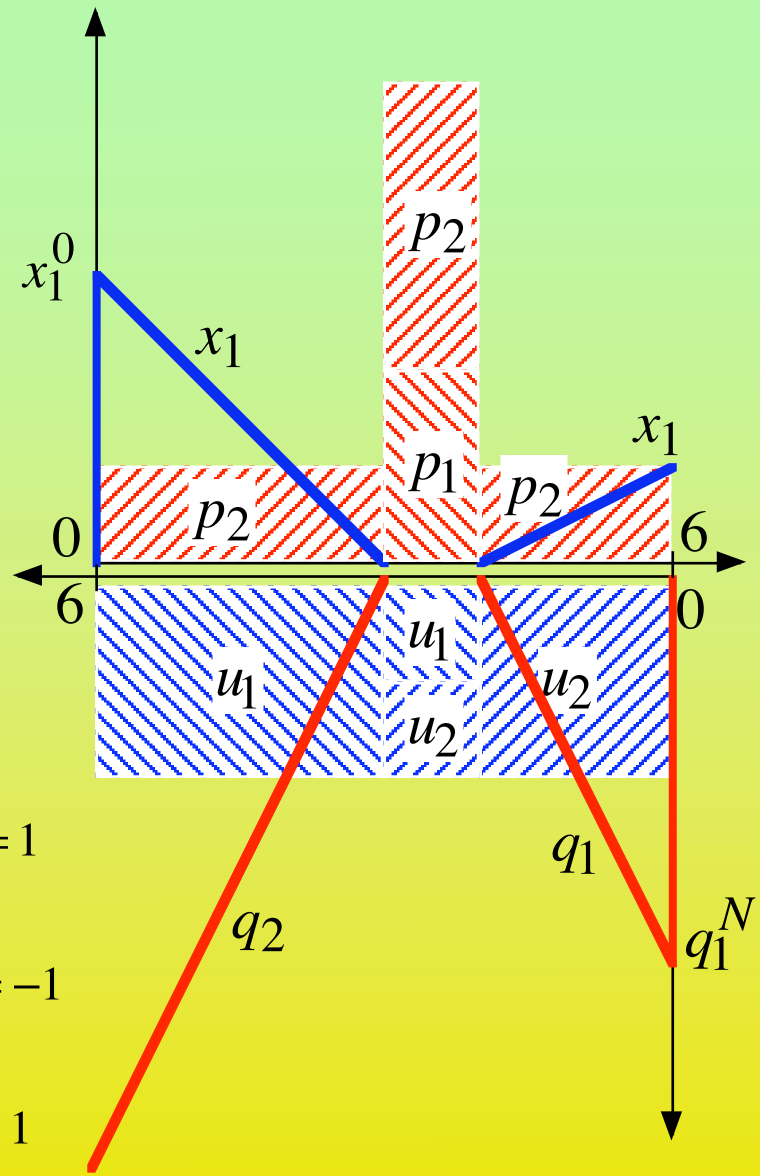
Boundary LP

$$\max \; -x_2^0$$
$$s.t. \quad x_1^0 + x_2^0 = 3$$
$$x^0 \ge 0$$

Solution $x_1^0 = 3$

$$\min \; 2q_2^N$$
$$s.t. \quad -q_1^N + q_2^N = -4$$
$$q^N \ge 0$$

Solution $q_1^N = 4$



Rates LP $2<T<5$

$$\max \; 2u_1 \quad - \dot{x}_2$$
$$s.t. \quad u_1 + \dot{x}_1 + \dot{x}_2 = 1$$
$$u_1 + u_2 = 2$$

$$\min \; p_1 \quad + 2\dot{q}_2$$
$$s.t. \quad p_1 - \dot{q}_1 + \dot{q}_2 = 2$$
$$p_1 - p_2 = -1$$

$\dot{x}_1$ **U**, u_1 **Z**, $\dot{x}_2, u_2$ **P**

Solution $u_2 = 2, \; \dot{x}_1 = 1$

$\dot{x}_1$ **U**, $\dot{x}_2, u_1, u_2$ **P**

Solution $u_1 = 2, \; \dot{x}_1 = -1$

$\dot{x}_1, \dot{x}_2, u_1, u_2$ **P**

Solution $u_1 = 1, \; u_2 = 1$

$\dot{q}_1$ **U**, p_1 **Z**, $p_2, \dot{q}_2$ **P**

Solution $p_2 = 1, \; \dot{q}_1 = -2$

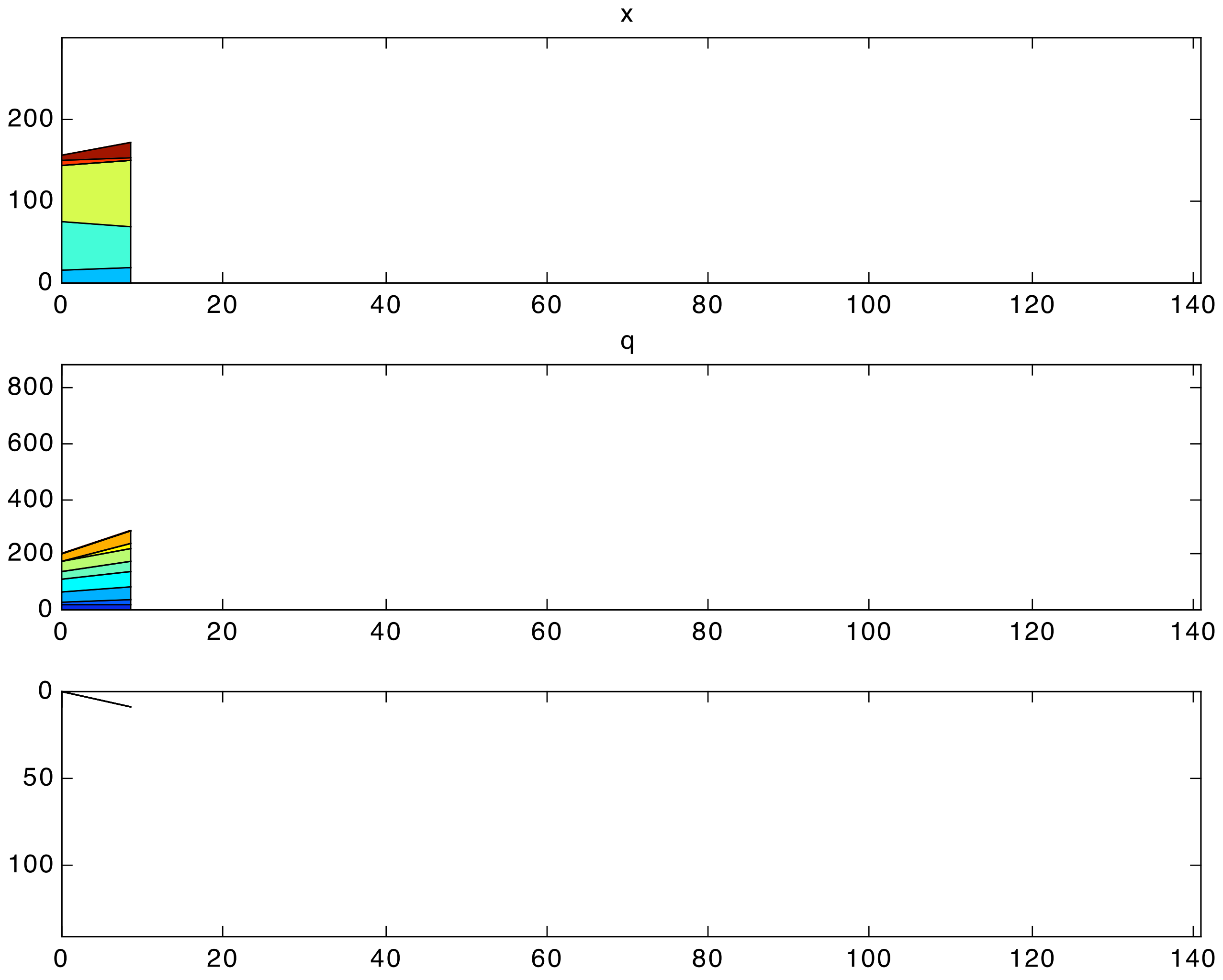
p_1 **Z**, $\dot{q}_1, p_2, \dot{q}_2$ **P**

Solution $p_2 = 1, \; \dot{q}_2 = 2$

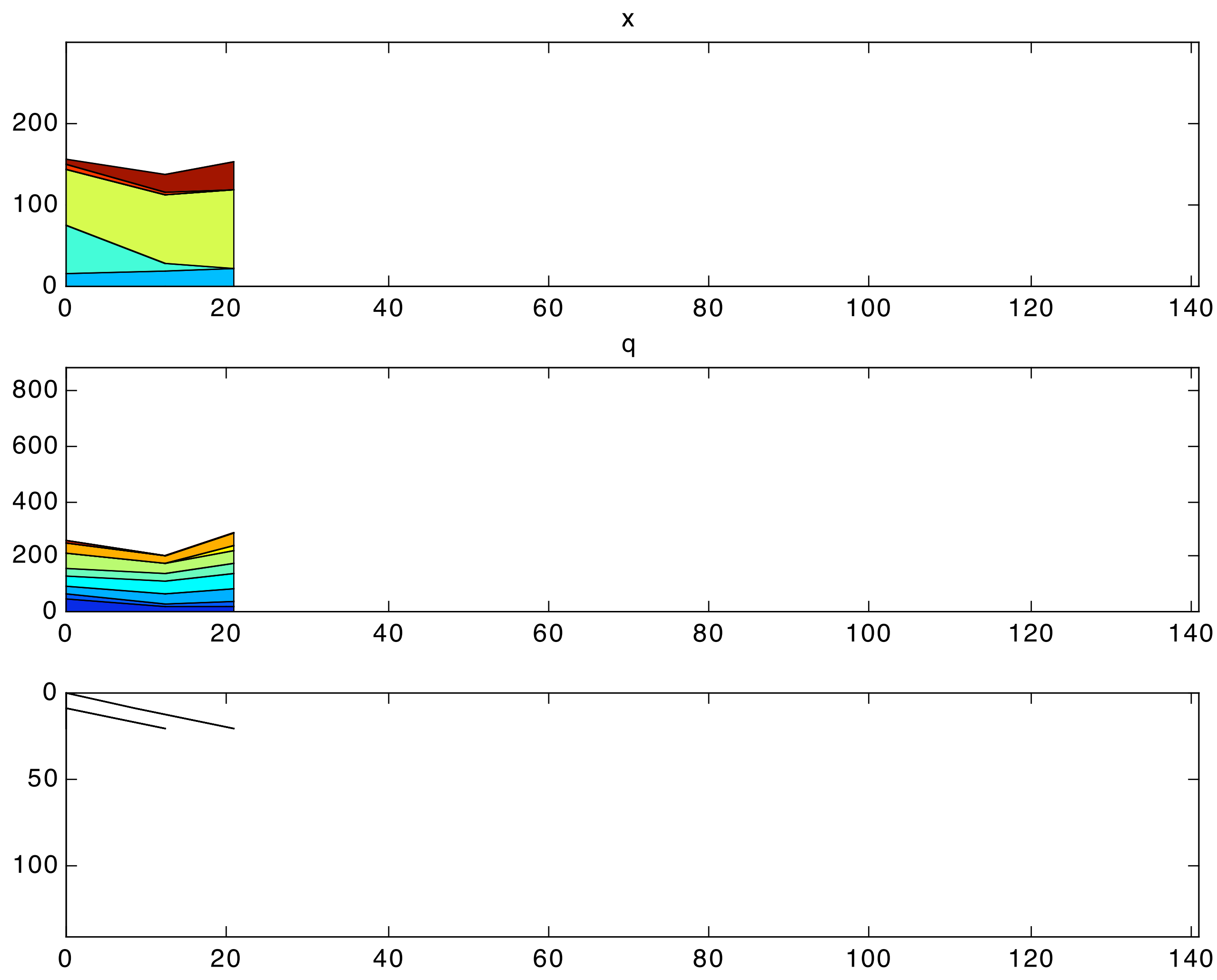
$p_1, \dot{q}_1, p_2, \dot{q}_2$ **P**

Solution $p_1 = 2, \; p_2 = 3$

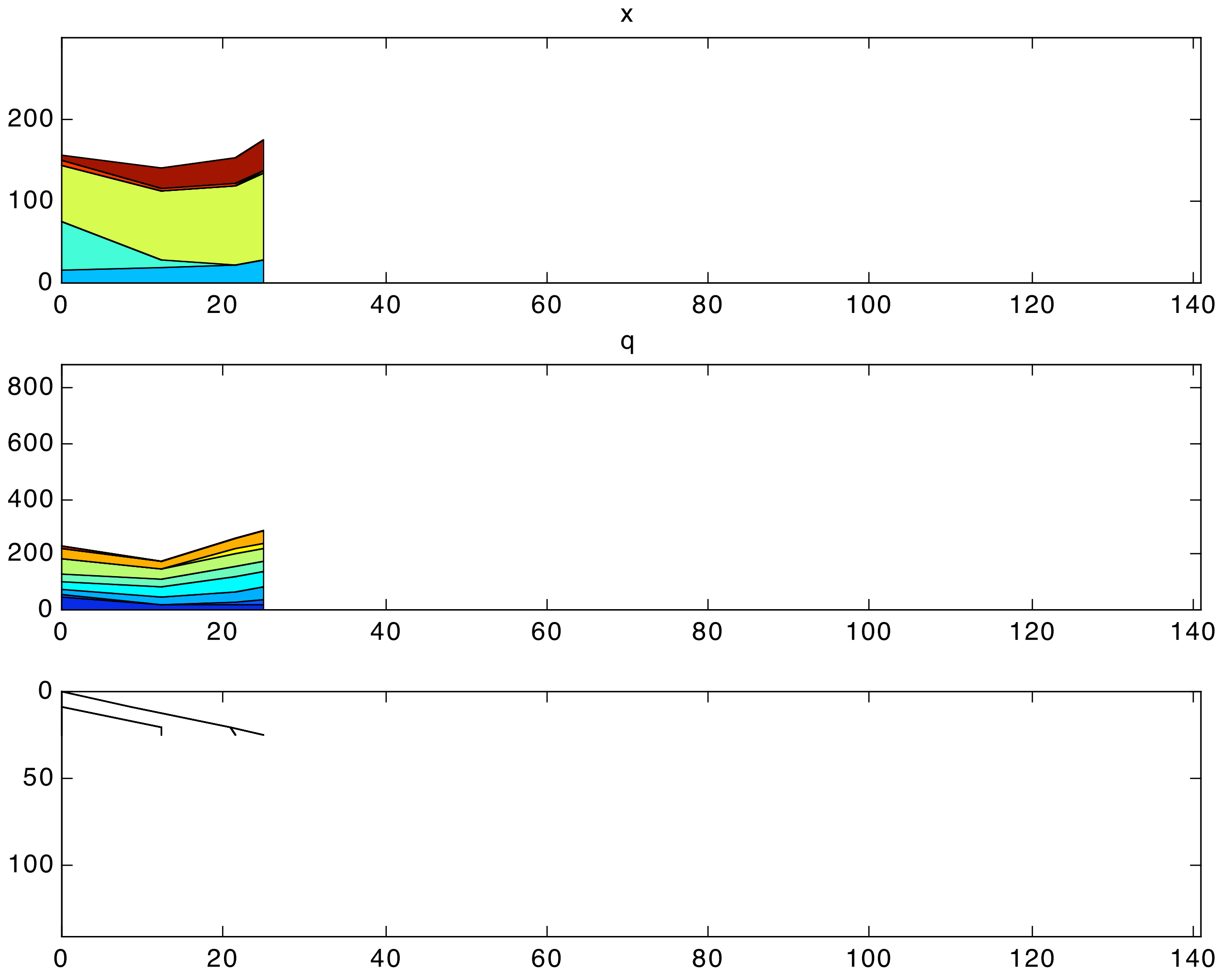
SCLP algorithm demo



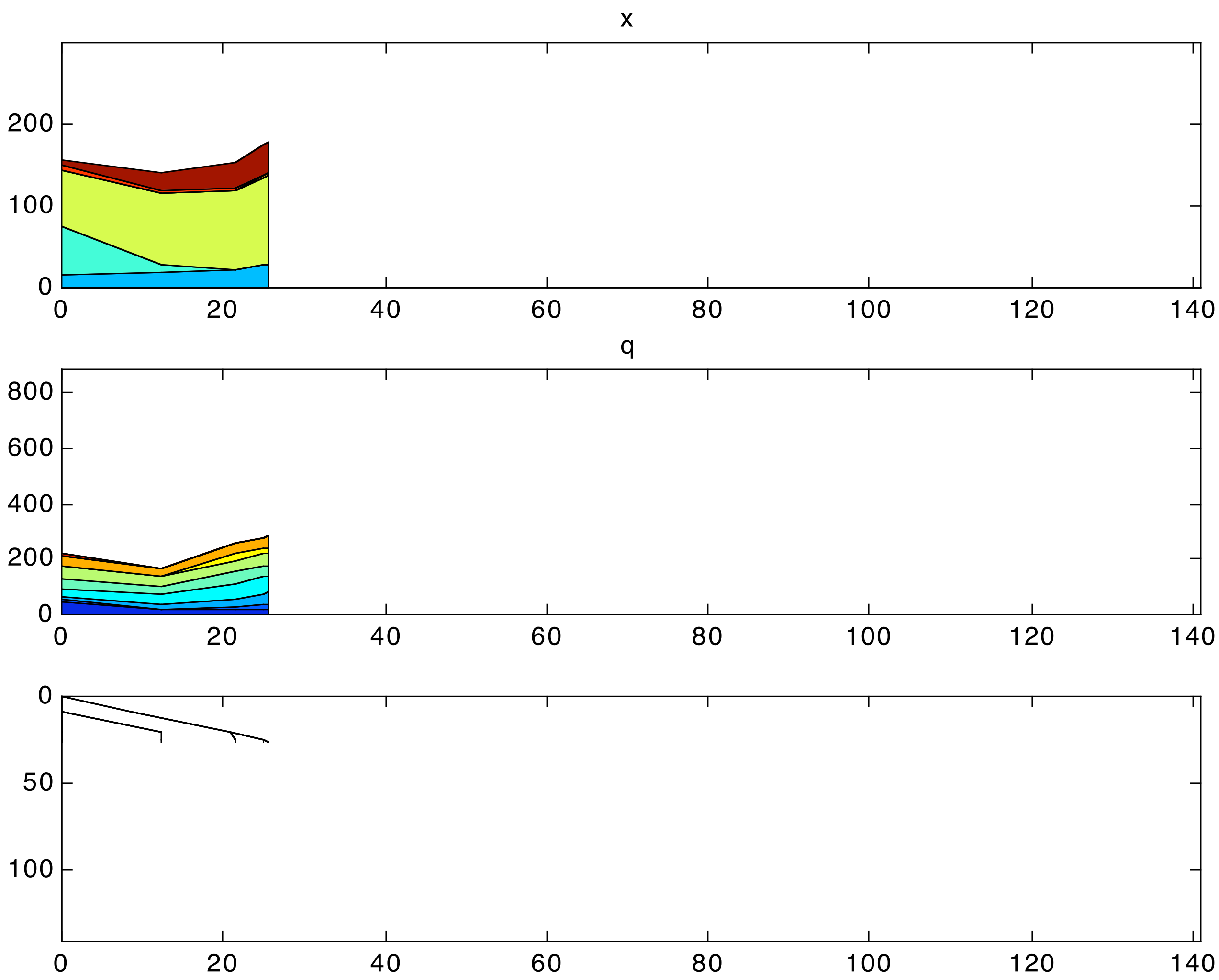
SCLP algorithm demo



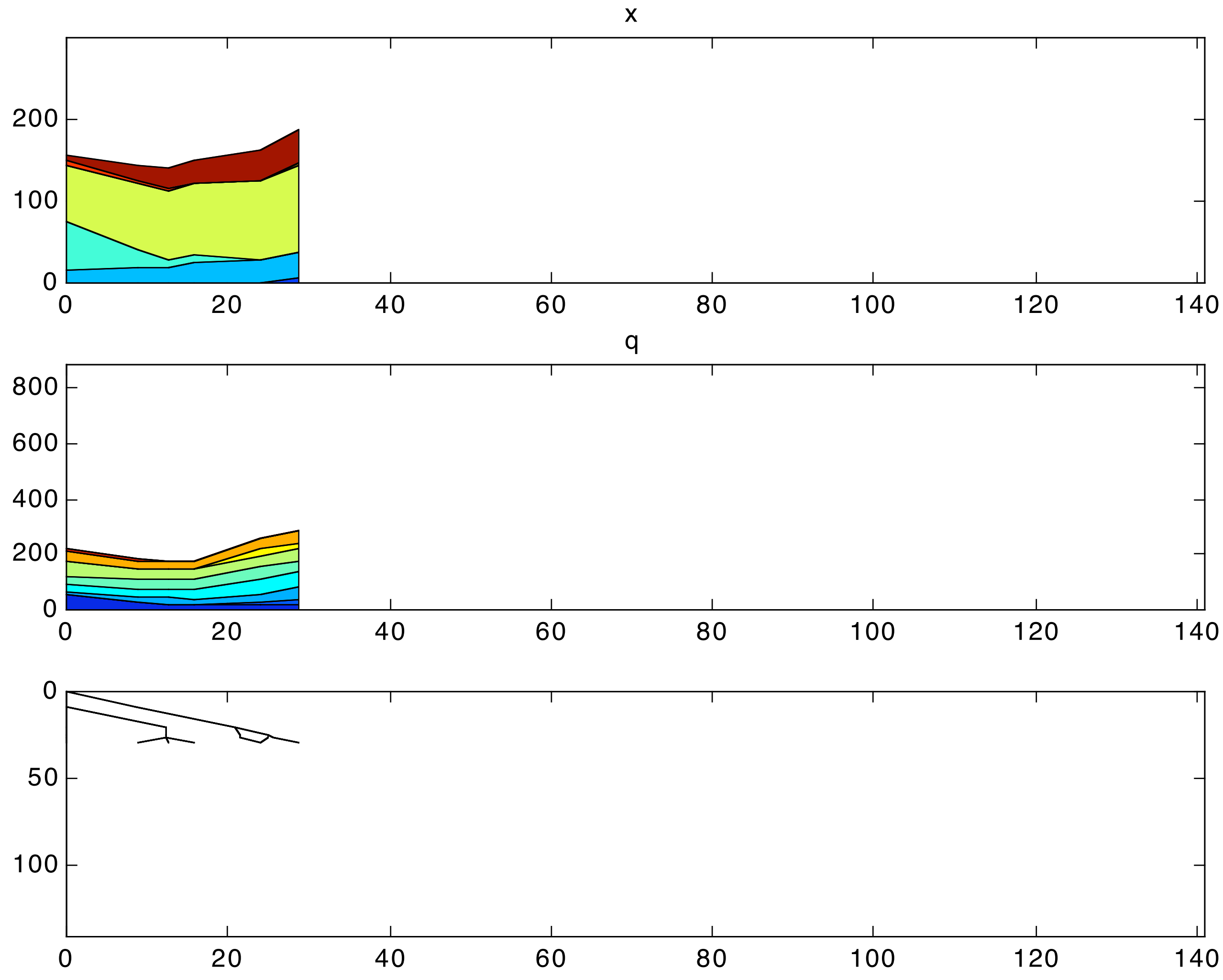
SCLP algorithm demo



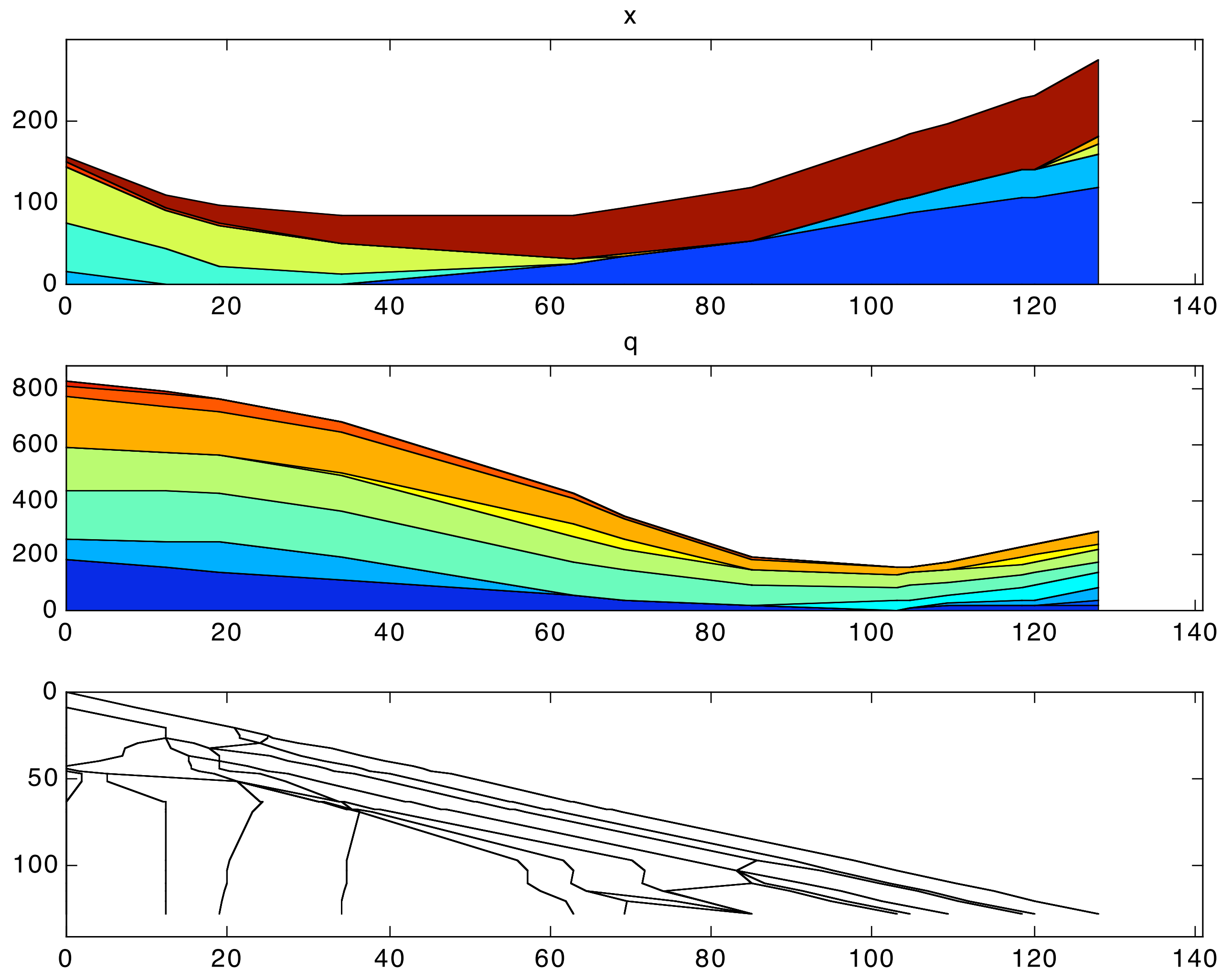
SCLP algorithm demo



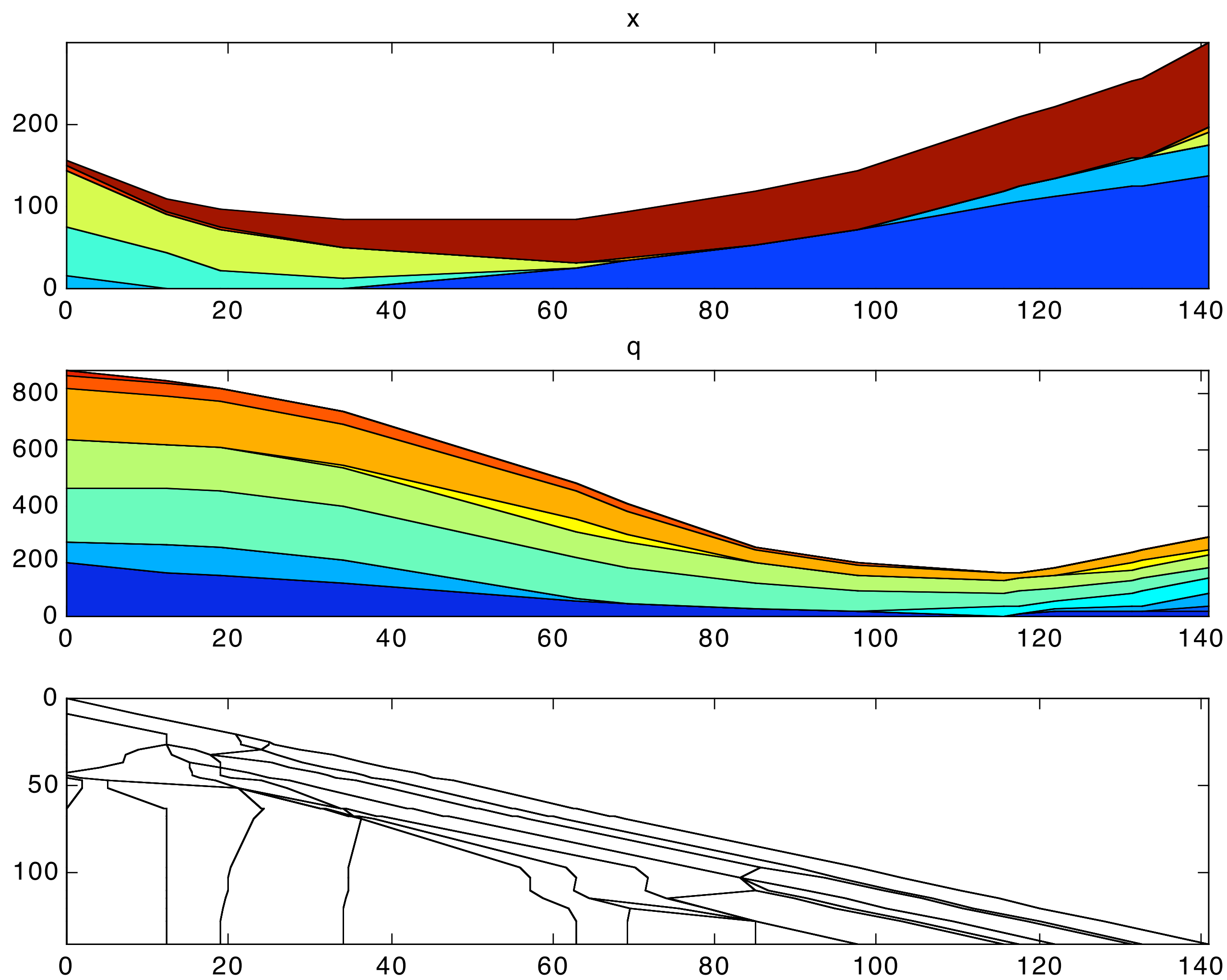
SCLP algorithm demo



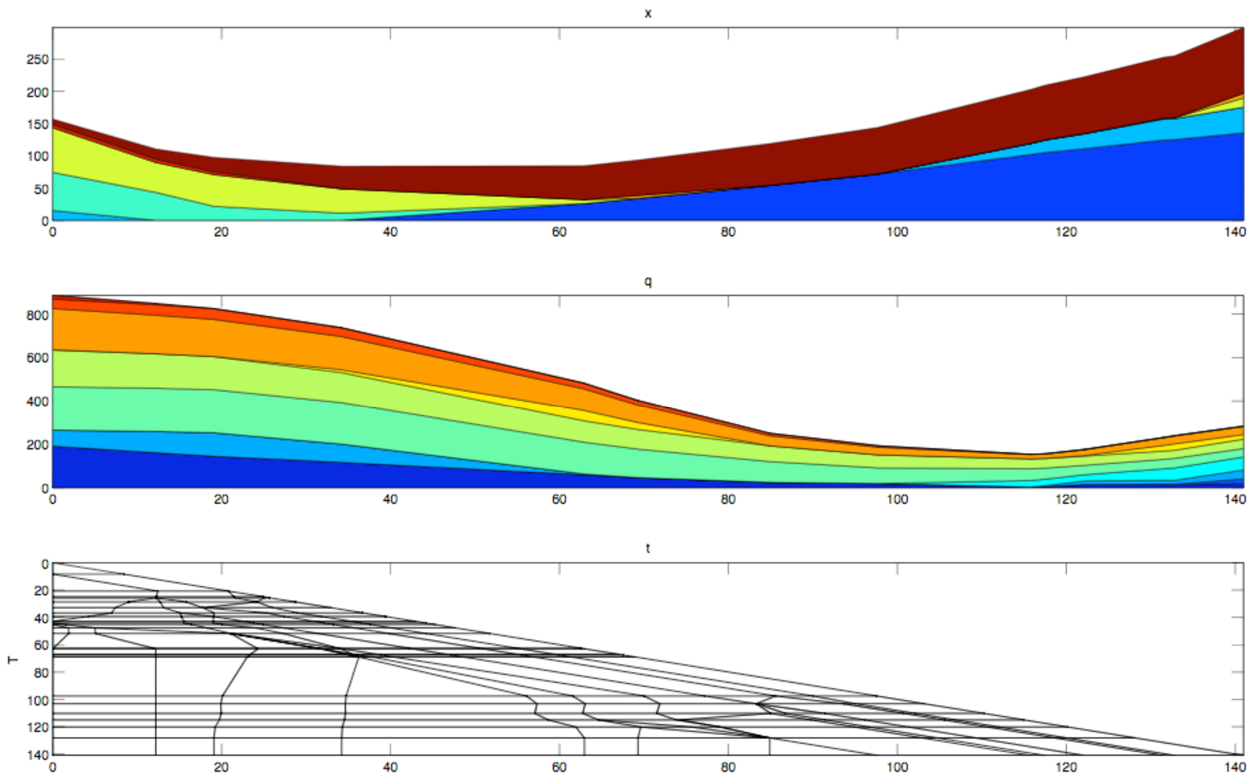
SCLP algorithm demo



SCLP algorithm demo



SCLP algorithm demo



Finite horizon control of multi class queueing networks MCQN

Control MCQN $(Q(t),T(t))$, over $0<t<T$

$$Q_k(t) = Q_k(0) - S_k(T_k(t)) + \sum_{k'} \Phi_{k'k}(S_{k'}(T_{k'}(t))) \quad k \in K$$

objective $$\min \sum_k \gamma_k T_k(T) + c_k \int_0^T Q_k(t)dt$$

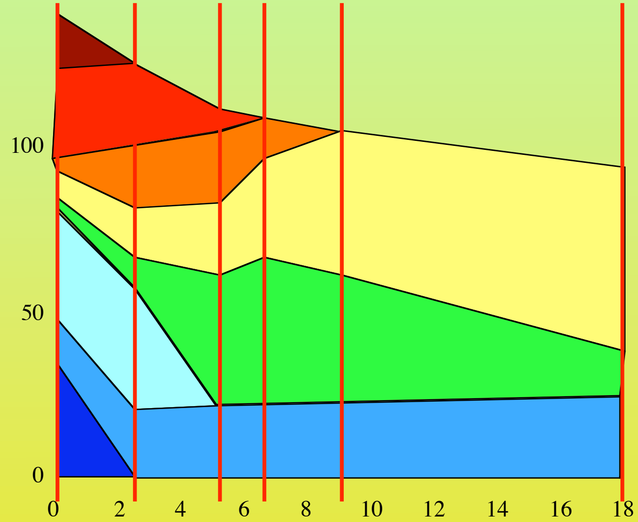
Fluid buffers can have 0 fluid
with positive flow
In real system: Standard Qs $k \in K_0$

Or they have fluid in whole interval:
In real system: IVQs $k \in K_\infty$

IVQs have nominal outflow rates,
from fluid solution:

$$\alpha_k = u_k : \quad k \in K_\infty$$

Optimal Fluid Solution

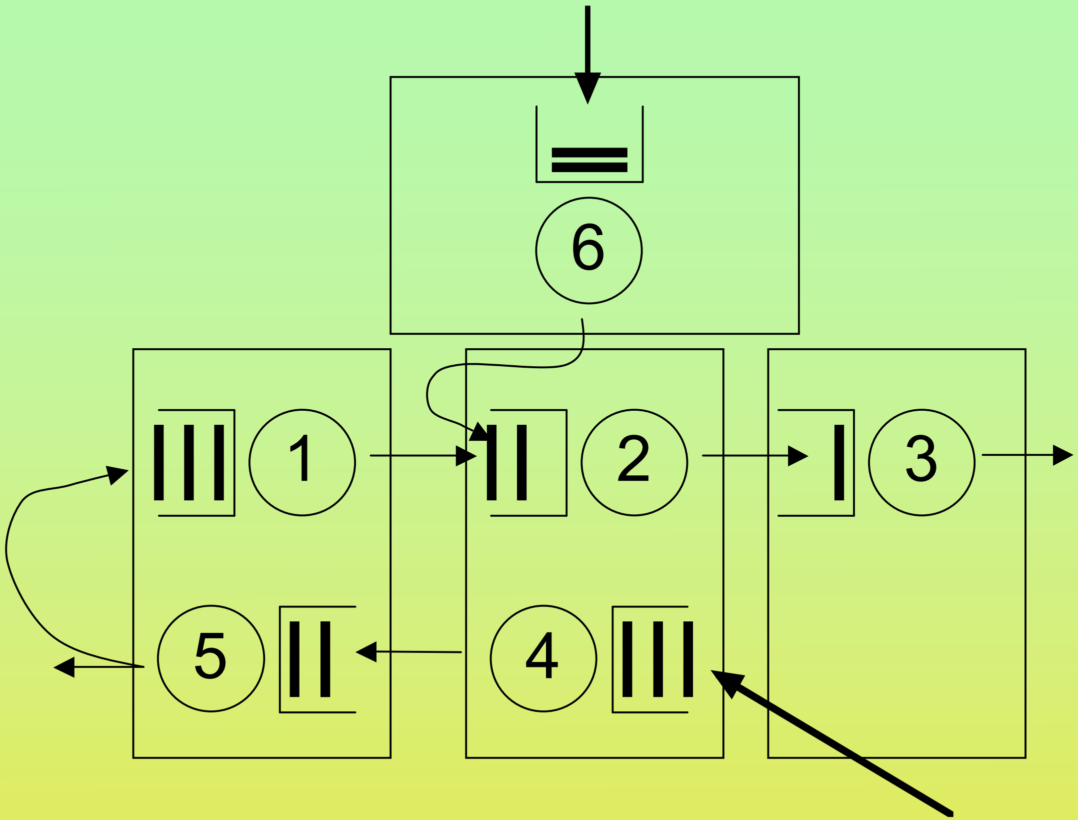


Tracking of Fluid:

- Model Deviations from Fluid as MCQN with IVQ
- Use Max Pressure to keep Deviations Stable

Asymptotically Optimal

Multi Class Queueing Networks - MCQN (Harrison 88, Dai 94, ...



$$Q_k(t) = Q_k(0) + A_k(t) - S_k(T_k(t)) + \sum_{k' \in K} \Phi_{k'k}(S_{k'}(T_{k'}(t)))$$

MCQN with Infinite Virtual Queues (Massey 84, 'W 06)

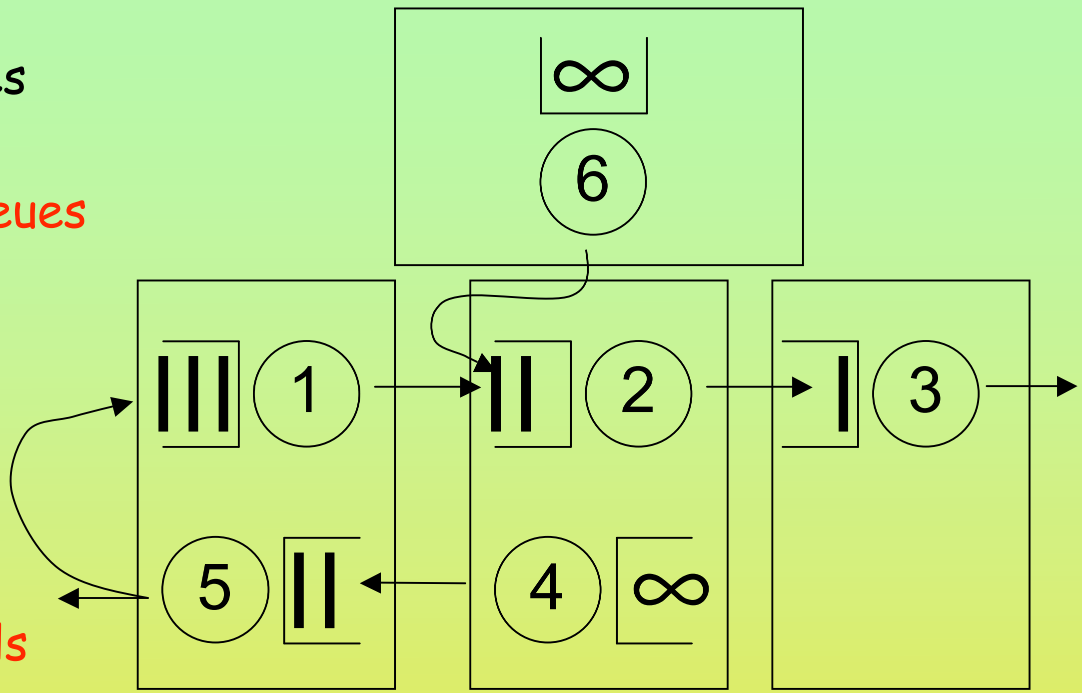
Two types of classes

Standard Queues

Infinite Virtual Queues

$K_0 = \{1,2,3,5\}$

$K_\infty = \{4,6\}$



No exogenous arrivals

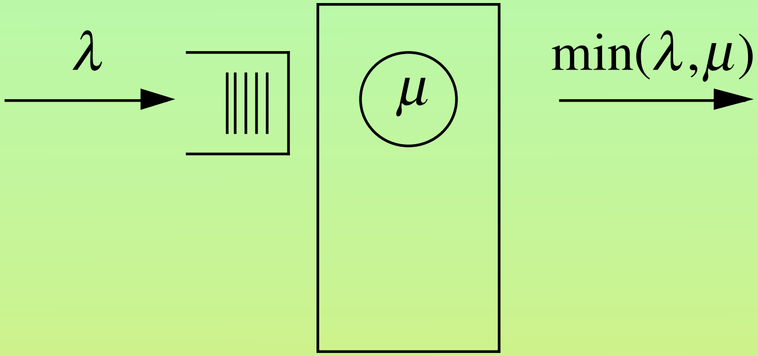
Nominal Input Rate

$$k \in K_0 : \qquad Q_k(t) = Q_k(0) - S_k(T_k(t)) + \sum_{k' \in \mathrm{K}} \Phi_{k'k}(S_{k'}(T_{k'}(t))) \geq 0$$

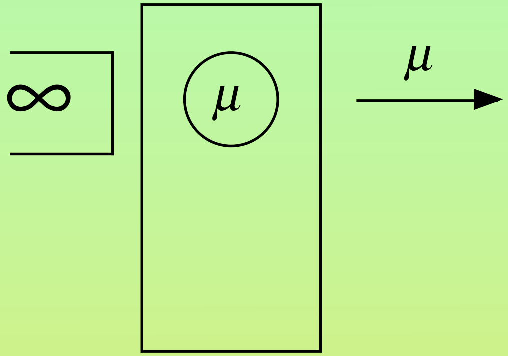
$$k \in K_\infty : \qquad Q_k(t) = Q_k(0) + \alpha_k t - S_k(T_k(t))$$

Infinite Virtual Queues –- Queueing vs Manufacturing

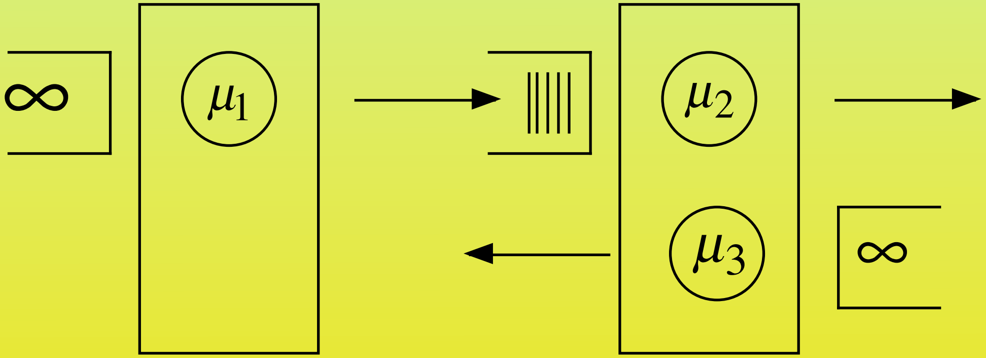
Single server queue



Machine with controlled input

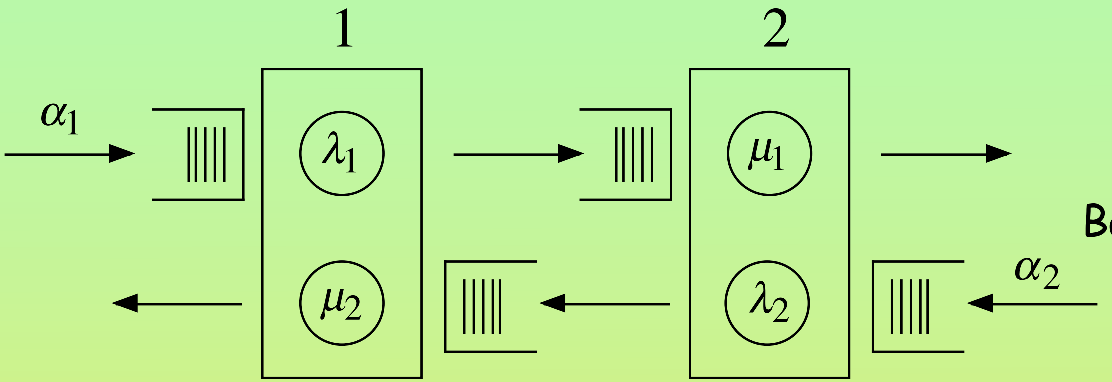


A tandem of queues



A new paradigm for balanced heavy traffic

Kumar-Seidman Rybko-Stolyar network (KSRS)

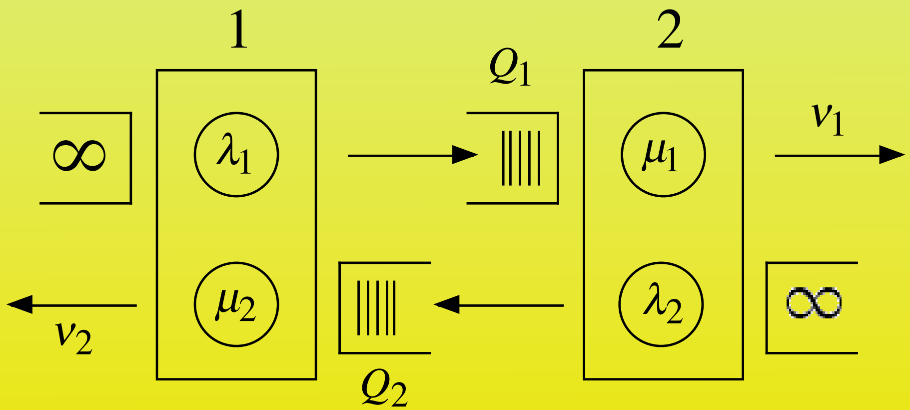


Heavy traffic when $\alpha_1 \nearrow, \alpha_2 \nearrow$

Balanced heavy traffic $\alpha_1 \to \nu_1, \alpha_2 \to \nu_2$

Push-Pull network - full utilization, $\rho=1$, no congestion

Both machines work non-stop, and no flow accumulates implies:



$$\nu_1 = \alpha_1\lambda_1 = (1-\alpha_2)\mu_1 = \frac{\lambda_1\mu_1(\mu_2-\lambda_2)}{\mu_1\mu_2-\lambda_1\lambda_2}$$

$$\nu_2 = \alpha_2\lambda_2 = (1-\alpha_1)\mu_2 = \frac{\lambda_2\mu_2(\mu_1-\lambda_1)}{\mu_1\mu_2-\lambda_1\lambda_2}$$

Finite horizon control of multi class queueing networks MCQN

Control MCQN $(Q(t),T(t))$, over $0<t<T$

$$Q_k(t) = Q_k(0) - S_k(T_k(t)) + \sum_{k'} \Phi_{k'k}(S_{k'}(T_{k'}(t))) \quad k \in K$$

objective $$\min \sum_k \gamma_k T_k(T) + c_k \int_0^T Q_k(t)dt$$

Fluid buffers can have 0 fluid
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In real system: Standard Qs $k \in K_0$

Or they have fluid in whole interval:
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IVQs have nominal outflow rates,
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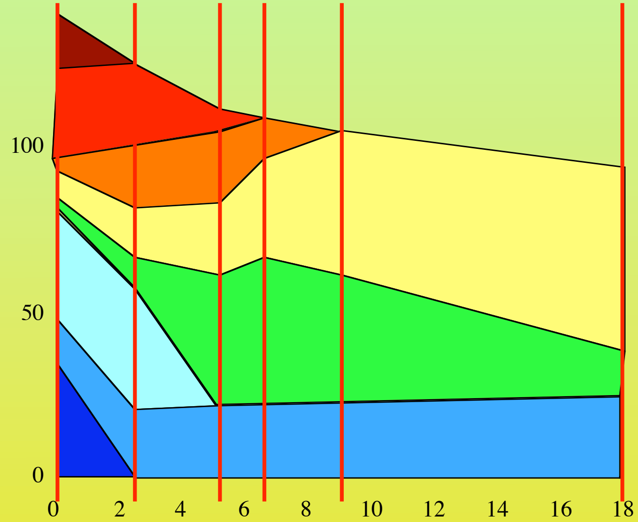
$$\alpha_k = u_k : \quad k \in K_\infty$$

Optimal Fluid Solution

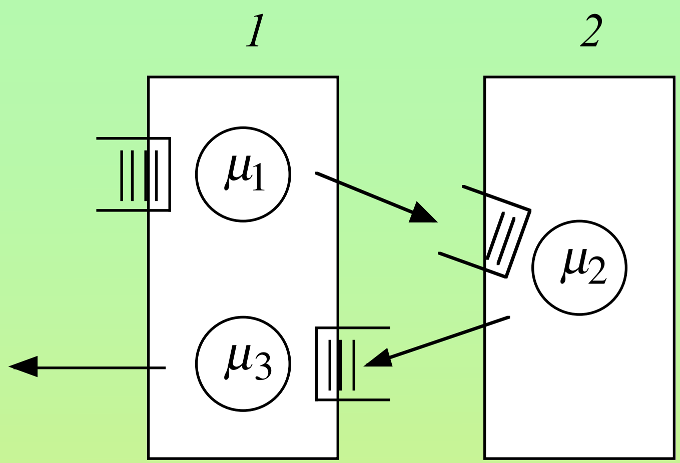


Tracking of Fluid:

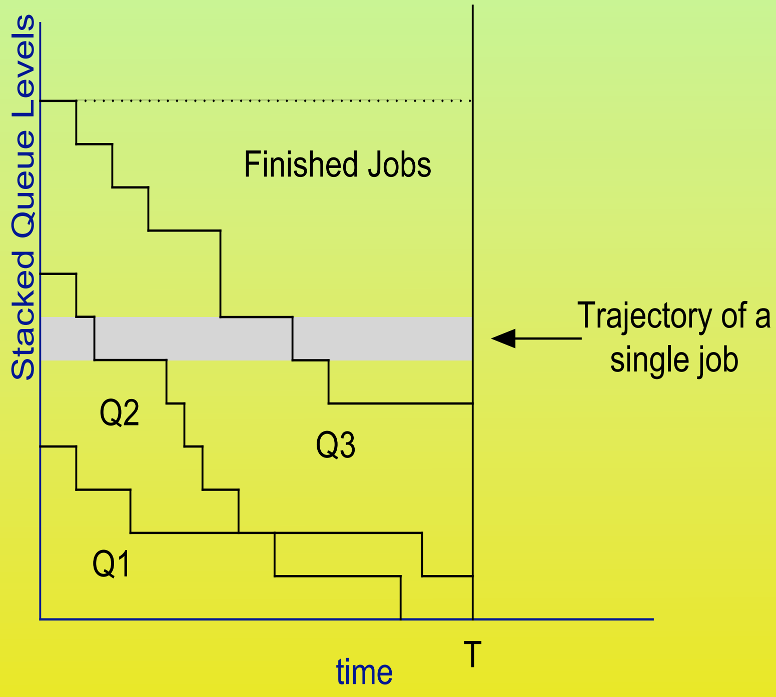
- Model Deviations from Fluid as MCQN with IVQ
- Use Max Pressure to keep Deviations Stable

Asymptotically Optimal

Example Network



Stacked Queue level representation:



Resource 2 is the bottleneck

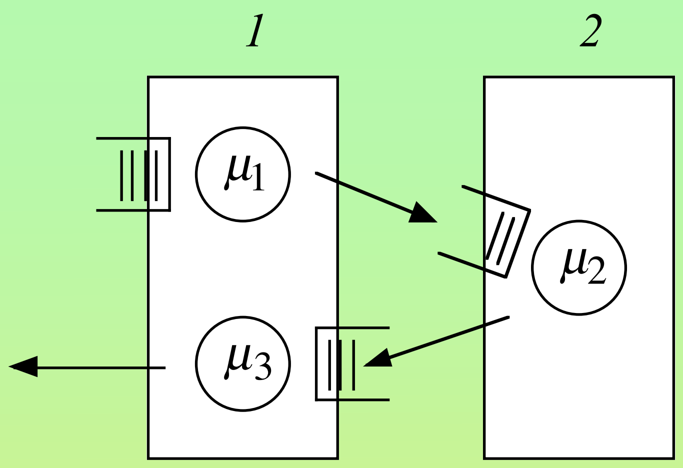
$$\frac{1}{\mu_2} > \frac{1}{\mu_1} + \frac{1}{\mu_3}$$

How to control queues 1,3

Schedule resource 1,

for optimal draining

Fluid formulation



$$\min \int_0^T q_1(t) + q_2(t) + q_3(t)dt$$

$$s.t: \quad q_1(t) = q_1(0) - \int_0^t \mu_1 u_1(s)ds$$

$$q_2(t) = q_2(0) + \int_0^t \mu_1 u_1(s)ds - \int_0^t \mu_2 u_2(s)ds$$

$$q_3(t) = q_3(0) + \int_0^t \mu_2 u_2(s)ds - \int_0^t \mu_3 u_3(s)ds$$

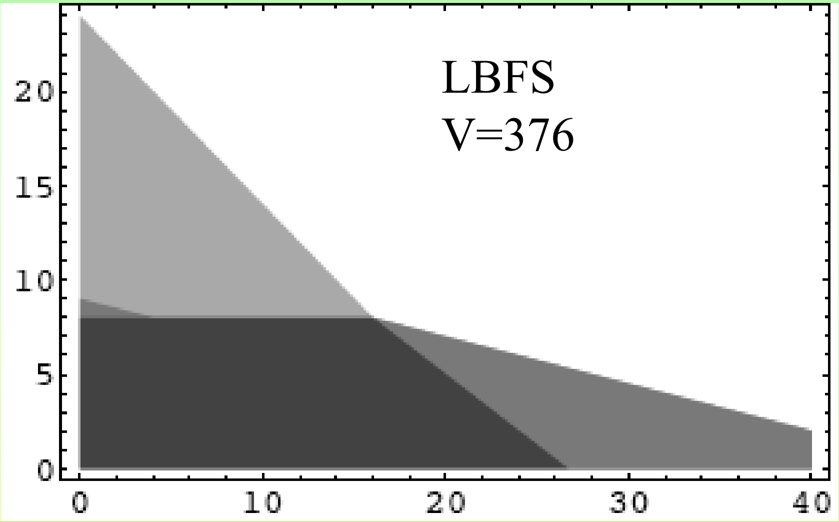
$$u_1(t) \quad + \quad u_3(t) \le 1$$

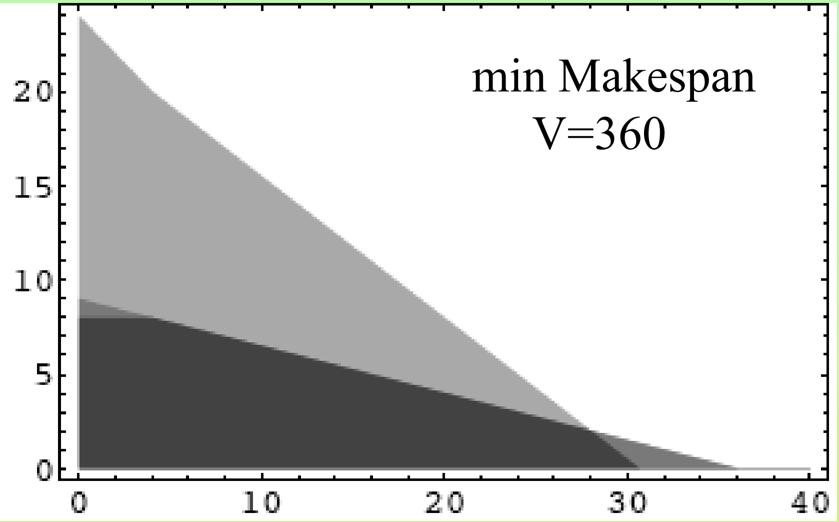
$$u_2(t) \le 1$$

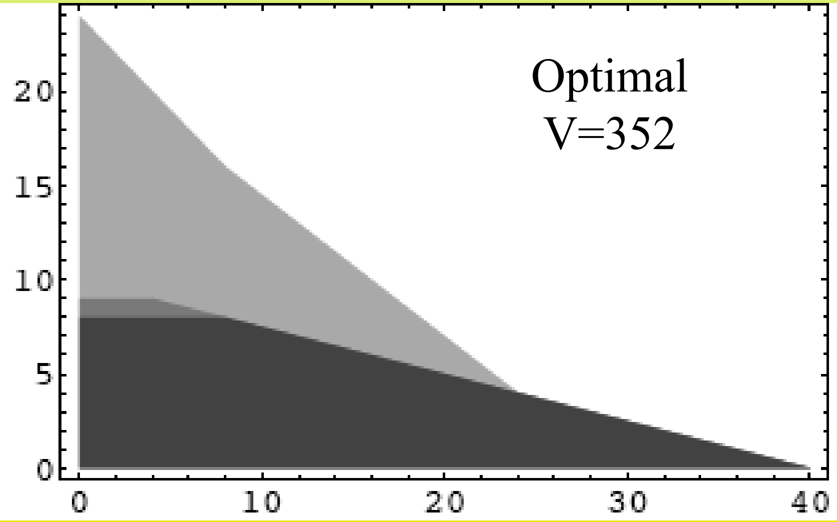
$$u,q \ge 0 \qquad t \in [0,T]$$

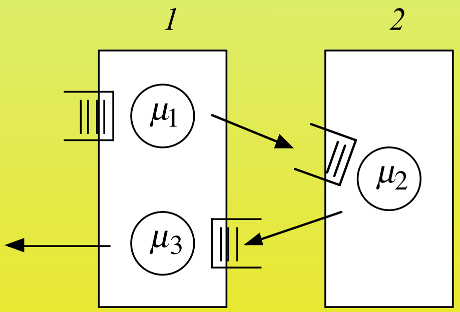
This is a Separated Continuous Linear Program (SCLP)

Fluid policies: LBFS, min Makespan, Optimal



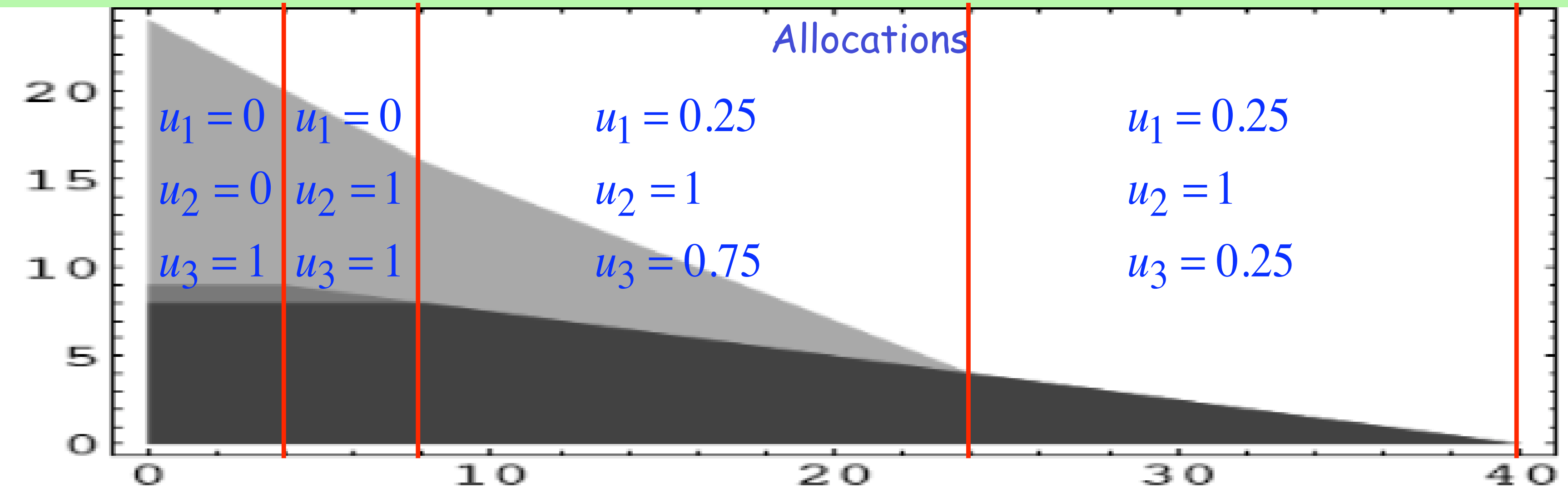






Time intervals of Optimal Fluid Solution

$\tau_1 = [0,4)$ $\tau_2 = [4,8)$ $\tau_3 = [8,24)$ $\tau_4 = [24,40)$

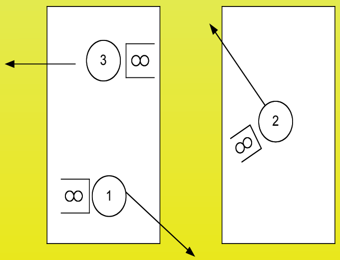


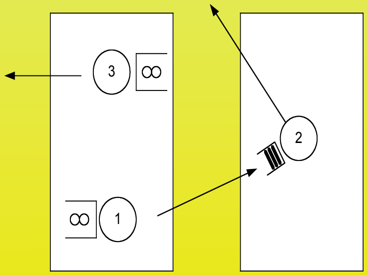
τ_1	τ_2	τ_3	τ_4
$u_1 = 0$	$u_1 = 0$	$u_1 = 0.25$	$u_1 = 0.25$
$u_2 = 0$	$u_2 = 1$	$u_2 = 1$	$u_2 = 1$
$u_3 = 1$	$u_3 = 1$	$u_3 = 0.75$	$u_3 = 0.25$

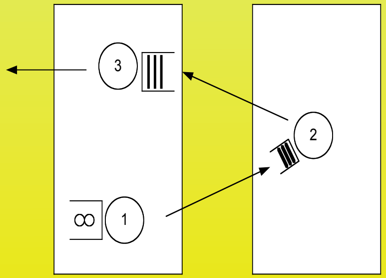
Define:

$$K_0^{\,n} = \{k \mid q_k(t) = 0,\ \ t \in \tau_n\}$$

$$K_\infty^{\,n} = \{k \mid q_k(t) \ge 0,\ \ t \in \tau_n\}$$







Maximum pressure policy (Tassiulas, Stolyar, Dai, Lin, Ata)

Max pressure achieves stability for any system with offered load < 1

rate stability if offered load $= 1$

Consider MCQN with fluid dynamics described by

$$\frac{d}{dt}Q(t) = \alpha - Ru(t)$$

where R is the input output matrix, $u(t)$ is the machine allocation, and α is the input rate.

The machine allocations (controls) are subject to resource constraints

Max pressure attempts to maximize the gradient

of the sum of squares of queue lengths

$$\frac{d}{dt}\sum_k Q_k^2(t) = \frac{d}{dt}Q(t)\bullet Q(t) = 2Q(t)\bullet(\alpha - Ru(t))$$

At any time t choose allocation $u(t)$ such that

$$\max Q(t)\bullet Ru(t) \quad \text{s.t.} \quad Au(t) \le \mathbf{1},\ u(t) \ge 0,\ u(t) \text{ is available}$$

In balanced heavy traffic it may optimize the diffusion approximation

Maximum Pressure (Tassiulas, Stolyar, Dai-Lin) for MCQN with IVQs

Theorem (Dai and Lin 2005):

MCQN under Max Pressure, when ρ≤1 is rate stable: $\lim_{t\to\infty} \frac{Q_k(t)}{t} = 0$

Adaptation to MCQN with IVQs

Theorem: Same holds for MCQN with IVQ,

$$k \in K_0 : \qquad Q_k(t) = Q_k(0) - S_k(T_k(t)) + \sum_{k' \in \mathrm{K}} \Phi_{k'k}(S_{k'}(T_{k'}(t))) \ge 0$$

$$k \in K_\infty : \qquad Q_k(t) = Q_k(0) + \alpha_k t - S_k(T_k(t))$$

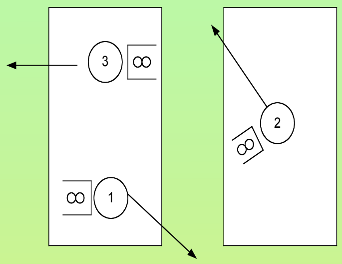
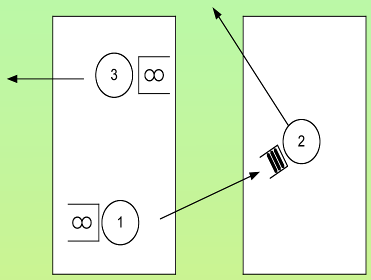
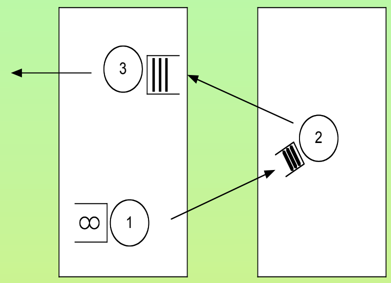
At any time t choose allocation $u(t)$ such that

$$\max Q(t) \bullet R u(t) \quad \text{s.t.} \quad Au(t) \le \mathbf{1}, \; u(t) \ge 0, \; u(t) \text{ is available} \qquad u(t) = \dot{T}(t)$$

α_k come from the fluid solution

$Q_k(t)$ Measure deviations from fluid solution

Input-Output matrixes when Infinite Virtual Queues exist

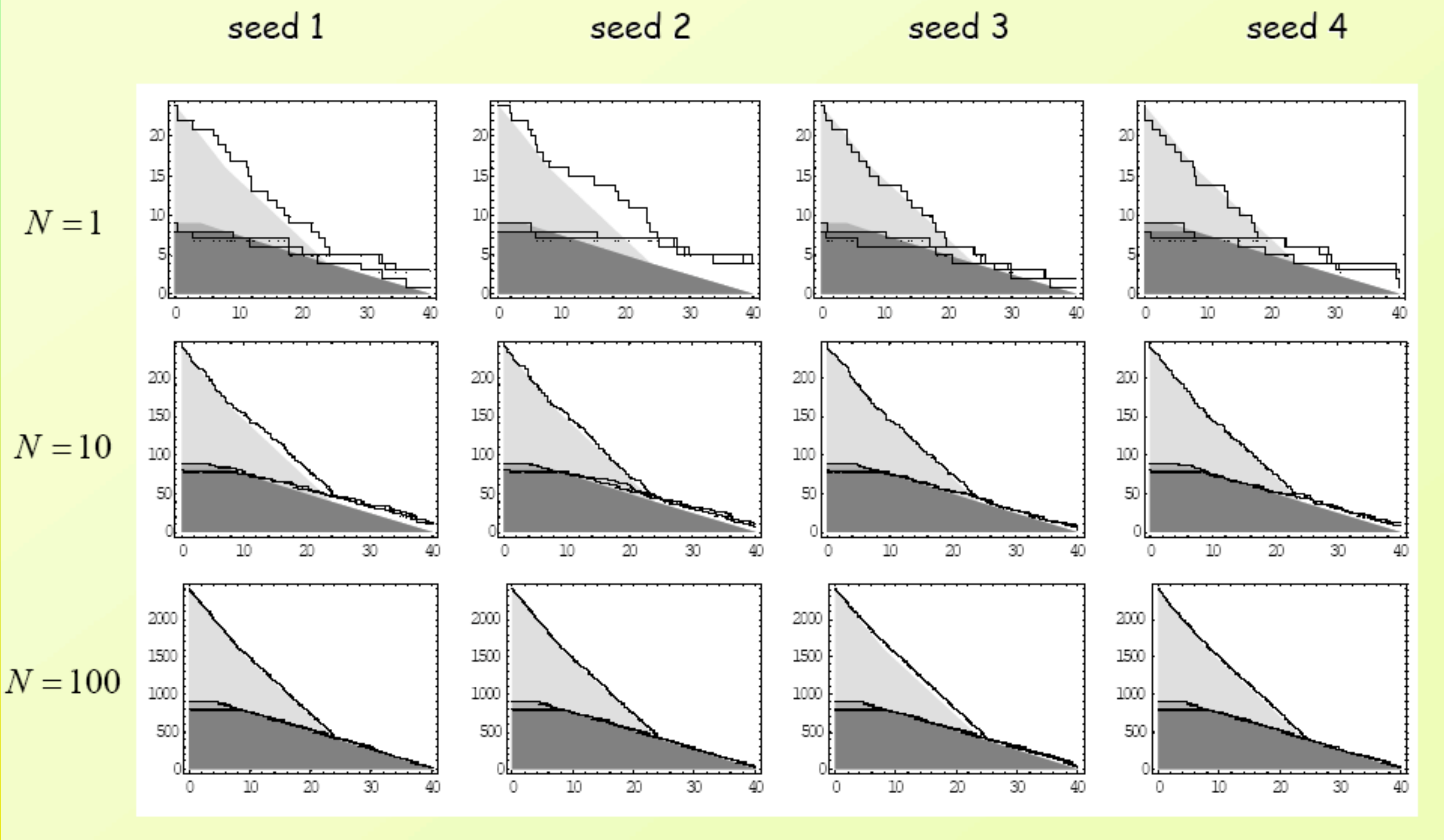
Time intervals 1 and 2	Time interval 3	Time interval 4
		
$R^1 = R^2 = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$	$R^3 = \begin{pmatrix} \mu_1 & 0 & 0 \\ -\mu_1 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$	$R^4 = \begin{pmatrix} \mu_1 & 0 & 0 \\ -\mu_1 & \mu_2 & 0 \\ 0 & -\mu_2 & \mu_3 \end{pmatrix}$

Implementing Maximum Pressure

Pressure for buffer k based on k and k+1

current \ downstream	$k+1 \in K_0$	$k+1 \in K_\infty$
$k \in K_0$	$\mu_k(Q_k - Q_{k+1})$	$\mu_k Q_k$
$k \in K_\infty$	$\mu_k(\alpha_k t - D_k - Q_{k+1})$	$\mu_k(\alpha_k t - D_k)$

Using Max Pressure to track fluid solution by MCQN-IVQ



Asymptotic Optimality

Theorem: Let Q(t) be the queue length process of a finite horizon MCQN.

Let $Q^N(t)$ be N scalings of Q(t), with $Q^N(0)=NQ(0)$, and with N-fold speed Of processing.

Let q(t) be the optimal fluid solution and let V_f be its Objective value.

(i) Let V^N denote the objective values of $Q^N(t)$ for any general policy. Then:

$$\liminf_{N\to\infty} \frac{1}{N} V^N \geq V_f \quad a.s.$$

(ii) Under max pressure tracking of the fluid solution

$$\lim_{N\to\infty} \frac{1}{N} Q(t)^N = q(t) \quad a.s. \text{ uniformly on } 0 < t < T$$

and

$$\lim_{N\to\infty} \frac{1}{N} V^N = V_f \quad a.s.$$

Summary

We considered control of MCQN over a finite time horizon

We formulated a fluid approximation and formulated SCLP

We described a simplex type algorithm to solve SCLP

Solution of SCLP provides optimal fluid solution

We Introduced MCQN with IVQs

IVQs add another level of control to MCQN

IVQs provide a new paradigm for balanced heavy traffic.

We fitted a MCQN with IVQ model to each interval of the fluid solution

We adapted Max Pressure policy to track fluid solution

This provides asymptotically optimal policy

References

(1) "A Simplex based algorithm to solve SCLP" W', Math Prog A, 115:151-198, 2008.

(2) "Near optimal control of queueing networks over a finite time horizon" Yoni Nazarathy and Gideon Weiss, Annals of OR online

(3) "A Push-Pull network with infinite supply of work" Anat Kopzon, Yoni Nazarathy and Gideon Weiss, Under review, QUESTA

(4) "Positive Harris recurrent and diffusion scale analysis of a Push-Pull queueing network with infinite supply of work" Yoni Nazarathy and Gideon Weiss, To be presented at Value Tools 2008, Performance Evaluation, to appear

Challenges

- Can we track fluid solution so as to be optimal on the diffusion scale ?

- Can we extend the policy for Push-Pull to general networks with infinite supply at all heavy traffic nodes ?